Dynamics of antimembranes in the maximally supersymmetric eleven-dimensional pp wave

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ABSTRACT: We study a spherical antimembrane in the eleven dimensional *pp* wave. In this background, a single antimembrane breaks all the supersymmetries because its dipole is misaligned with the background flux. Using the BMN matrix theory we compute the one-loop potential for the antimembrane. Then we put the antimembrane in the field produced by a source spherical membrane and compute the velocity-dependent part of the interaction between them on both the supergravity side and the BMN matrix theory side. Despite the aforementioned nonsupersymmetry of the antimembrane, it is found that the results on the two sides completely agree.

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Contents

1. Introduction

Banks, Fischler, Shenker and Susskind proposed [1] that M-theory should be described nonperturbatively by what is now known as the BFSS matrix theory. It has passed many tests. For example, it reproduces eleven dimensional supergravity computations such as interactions between gravitons and other objects [1–9]. However, the study of BFSS matrix theory is not very easy because the flat directions in its potential result in a continuous spectrum. The situation is improved in the maximally supersymmetric eleven dimensional pp wave background for which [10] proposed the BMN matrix theory. The BMN theory

is a mass deformation of the BFSS theory.[1] The mass deformation (parameterized by μ, the strength of the background four-form) lifts the flat directions in the potential, giving a discrete spectrum with the vacua being concentric spherical membrane configurations. It is then natural to investigate in BMN matrix theory the dynamics of membranes [12, 13], as well as gauge/gravity dualities [14–16].

In [14, 15] the interactions between gravitons, and between spherical membranes, in the pp wave[2] were investigated, and it was shown that one-loop computations on the matrix theory side properly reproduce the results on the supergravity side. In this paper, we continue our work along this line and consider the dynamics of a spherical antimembrane in the pp wave background.

Of course, in the pp wave the spherical branes that we call membranes and antimembranes have no net membrane charge. The distinction between "membranes" and "antimembranes" is in their nonzero dipoles, which have opposite sign between the two. Upon taking the flat space limit, one looks at small regions, e.g. near the north poles of the spheres, where there is a local concentration of positive charge for the "membrane" and local concentration of negative charge for the "antimembrane"; these correspond to the usual membrane and antimembrane in flat space.

In flat space, an infinitely extending flat antimembrane is half BPS, just as a membrane is. In the pp wave, the nonvanishing background four-form field strength changes the situation. A spherical membrane (centered at the origin and of an appropriate size) is half BPS, while a spherical antimembrane breaks all the supersymmetry. This is because whereas the dipole of the membrane is "aligned" with the background flux, that of the antimembrane is "antialigned" with it.

Using the BMN matrix theory we compute the one-loop potential for the antimembrane. Then we put the antimembrane in the field produced by a source spherical membrane (on top of the pp wave background of course) and compute the velocity-dependent part of the interaction between them on both the supergravity side and the BMN matrix theory side. Although the antimembrane breaks all the supersymmetries, complete agreement is found between the results on the two sides.

In flat space, [18, 19] considered the interaction of a membrane-antimembrane pair (in the IIA language, a D2-$\overline{\text{D2}}$ pair with a large number N of D0's bound to each) and found that the results of computations in BFSS matrix theory and supergravity agree. In flat space, the brane and the antibrane are individually half BPS although the pair is nonsupersymmetric. Furthermore it can be argued [19] that, through a series of T and S dualities, the D2-$\overline{\text{D2}}$ system (with N D0 branes bound to each) can be mapped into two clusters of D0-branes moving at a small relative transverse velocity $v \sim \frac{1}{N}$ and is therefore approximately supersymmetric for large N; this D0 system is known [1] to exhibit agreement between matrix theory and supergravity results. The work presented in this paper can be regarded as the generalization of [18, 19] to the pp wave in some sense, although in the pp wave background there is a big difference because, as we have

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[1]For other Matrix theories in non-flat backgrounds, see e.g. [11]. However, in [11] the approximation of a weakly curved background was made, whereas the BMN matrix theory is an exact one.

[2]In this paper, "the" pp wave refers to the maximally supersymmetric eleven dimensional one. [17]

pointed out, the antimembrane itself breaks all the supersymmetry. In [14] it was shown that BMN matrix theory and supergravity agree on the interaction between two gravitons in the pp wave, and one might ask whether, by some analog of the duality transformations in flat space, the membrane-antimembrane pair in the pp wave can also be transformed into two clusters of D0-branes — more precisely, two clusters of M-gravitational waves[3] — moving at small relative speed, which would therefore be approximately supersymmetric, as in flat space. The complete agreement found in this paper for the velocity-dependent part of the interaction certainly suggests that this could indeed be the case. However, we still have to compute the velocity-independent part of the interaction to get the complete story. Moreover, in flat space, the spatial world volumes of the D2-branes are tori and one can perform T-duality; for the pp wave, the branes are spheres, and at best we do not understand how to carry out T-duality. Hence, whether such a duality transformation exists in the pp wave is not completely clear to us at this stage.

The paper is organized as follows. In section 2 we identify the antimembrane configuration in both supergravity and BMN matrix theory. In section 3 we compute the one-loop effective potentials on the BMN matrix theory side, first computing the potential for the antimembrane by itself, then computing its interaction with the membrane. In section 4 we compute the membrane-antimembrane interaction on the supergravity side and compare it with the matrix theory result. We end with a Discussion. The technical details of diagonalizing the fluctuating modes are given in the appendices.

2. Antimembranes in the eleven dimensional pp wave

In eleven dimensional supergravity, the antimembrane's lightcone lagrangian can be obtained from that of the membrane by replacing $A_{\mu\nu\rho}$ in the latter with $-A_{\mu\nu\rho}$. The lagrangian density of a membrane in a general background is[4]

$$\mathcal{L} = -T\left[\sqrt{-\det(g_{ij})} - \frac{1}{6}\epsilon^{ijk}A_{\mu\nu\rho}\partial_i X^\mu \partial_j X^\nu \partial_k X^\rho\right], \tag{2.1}$$

and the lagrangian density of an antimembrane (distinguished from that of a membrane by use of an overbar) is obtained by flipping the sign of the Wess-Zumino term, so that

$$\bar{\mathcal{L}} = -T\left[\sqrt{-\det(g_{ij})} + \frac{1}{6}\epsilon^{ijk}A_{\mu\nu\rho}\partial_i X^\mu \partial_j X^\nu \partial_k X^\rho\right]. \tag{2.2}$$

The lightcone lagrangian density, $\mathcal{L}_{\text{l.c.}}(X^A, \dot{X}^A, \partial_r X^A; X^-, \Pi_-, \partial_r X^-)$, is obtained through Legendre transformation of the x^- degree of freedom. As for the untransformed

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[3]We note the curious fact for pp waves that the compactification breaks precisely those supersymmetries preserved by the momentum along the circle so that D0 branes effectively break all the supersymmetry of the IIA background, although there is supersymmetry in 11-dimensions. This was first noticed in the context of the 26 supercharge pp wave in [20]. But at the end of the day, we are not interested in the compact theory, but only use the language of D-branes for convenience.

[4]We use indices $i, j, k, \cdots = 1, 2, 3$; $a, b, c, \cdots = 4, \ldots, 9$; and $I, J, K, \cdots = 1, \ldots, 9$, unless otherwise stated. Indices $\mu, \nu, \cdots = +, -, 1, \ldots, 9$ are 11-dimensional curved-space indices.

lagrangian, we flip the sign of $A_{\mu\nu\rho}$ in $\mathcal{L}_{\text{l.c.}}$ to get the lightcone lagrangian for the antimembrane. For the pp wave, the only nonvanishing component of the three-form is

$$A_{+ij} = \frac{\mu}{3}\epsilon_{ijk}x^k, \tag{2.3}$$

so equivalently we can replace ϵ_{ijk} with $-\epsilon_{ijk}$. Let us consider a spherical antimembrane with $X^i X^i = r_0'^{\,2}$, $X^a = 0$, and M-momentum density $\Pi_- = \sin\theta \tilde{p}^+$. After integrating over the sphere, its lightcone lagrangian is,

$$
\begin{aligned}
\bar{L}_{\text{l.c.}} &= 4\pi \left[\frac{\tilde{p}^+}{2}\left(\frac{dr_0'}{dt}\right)^2 - \frac{\tilde{p}^+}{18}\mu^2\left(r_0'\right)^2 - \frac{T^2}{2\tilde{p}^+}\left(r_0'\right)^4 - T\frac{\mu}{3}\left(r_0'\right)^3 \right], \\
&= 4\pi \frac{\mu^4(\tilde{p}^+)^3}{18T^2}\left[\left(\frac{1}{\mu}\frac{d\eta}{dt}\right)^2 - \frac{1}{9}\eta^2 - \frac{1}{9}\eta^4 - \frac{2}{9}\eta^3 \right], \qquad \eta \equiv \frac{3T}{\mu\tilde{p}^+}r_0'.
\end{aligned} \tag{2.4}
$$

Note that the potential $\bar{V}(\eta) \sim \left(\frac{1}{9}\eta^2 + \frac{1}{9}\eta^4 + \frac{2}{9}\eta^3\right) = \frac{1}{9}\eta^2(\eta+1)^2$ is a monotonically increasing function with its only minimum being at $\eta = 0$. (Contrast this with the membrane potential $V(\eta)$, for which the η^3 term has the opposite sign, and thus a local minimum at $\eta = 1$.) Because of the "wrong" sign of the dipole it carries, an antimembrane of constant size does not solve the equation of motion and is therefore an unstable configuration. This is in contrast to flat space, for which a single antimembrane is half BPS and thus stable.

In the BMN matrix theory, the above spherical antimembrane configuration is given by replacing X^i with $-X^i$ in the usual fuzzy-sphere solution. Let us see why this is so. The action of the matrix theory in a generic weakly curved eleven dimensional background [11, 21] contains the following term describing the coupling of the matrix theory object given by X^i with the background three-form, up to an overall numerical factor

$$S_1 = J^{MNP(i_1\ldots i_n)}(X^i)\partial_{i_1}\cdots\partial_{i_n}A_{MNP}(0), \tag{2.5}$$

$(M, N, P = 0, 1, \ldots, 10,\ i, i_1, \ldots, i_n = 1, \ldots, 9)$ where $J^{MNP(i_1\ldots i_n)}(X^i)$ is (the moment of) the three-form current, and $\partial_{i_1}\cdots\partial_{i_n}A_{MNP}(0)$ is (the derivative of) the background three-form, evaluated at the origin. For the pp wave, this gives

$$S_1 = J^{+ij(i_1)}\left[\partial_{i_1}\left(\frac{\mu}{3}\epsilon_{ijk}x^k\right)\right](0) = J^{+ij(k)}\frac{\mu}{3}\epsilon_{ijk} = -\frac{\mu}{18R}\,\text{Tr}\left(i\left[X^i, X^j\right]X^k\right)\epsilon_{ijk}, \tag{2.6}$$

where in the last line we've used the fact that [21]

$$J^{+ij(k)} = -\frac{1}{6R}\,\text{Tr}\left(i\left[X^i, X^j\right]X^k\right). \tag{2.7}$$

As one can see S_1 is nothing but the Myers term in the BMN matrix theory. Now sending X^i to $-X^i$ flips the sign of S_1, which corresponds to the sign-flip of the Wess-Zumino term on the supergravity side we mentioned earlier. Hence if X^i represents a spherical membrane, $\bar{X}^i \equiv -X^i$ will represent a spherical antimembrane.

3. Computation in the BMN matrix theory

3.1 Tree level

The BMN matrix theory action is [10]

$$
S = \int dt \, \mathrm{Tr} \left\{ \sum_{I=1}^{9} \frac{1}{2R} (D_t X^I)^2 + i\psi^T D_t \psi + \frac{(M^3 R)^2}{4R} \sum_{I,J=1}^{9} \left[X^I, X^J \right]^2 - \right.
$$
$$
- (M^3 R) \sum_{I=1}^{9} \Psi^\dagger \gamma^A \left[\Psi, X^I \right] + \frac{1}{2R} \left[-\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3} (X^i)^2 - \left(\frac{\mu}{6}\right)^2 \sum_{a=4}^{9} (X^a)^2 \right] -
$$
$$
\left. - i\frac{\mu}{4} \Psi^\dagger \gamma_{123} \Psi - i\frac{(M^3 R)\mu}{3R} \sum_{i,j,k=1}^{3} \epsilon_{ijk} X^i X^j X^k \right\}, \tag{3.1}
$$

where $D_t X^I = \partial_t X^I - i\left[X_0, X^I\right]$, M is the eleven dimensional Planck mass, and R is the radius of the M-circle. We also define the parameter $\alpha = \frac{1}{M^3 R}$. In what follows we set both M and R to one (and hence $\alpha = 1$), which can be easily restored later. The background field configuration is

$$
B^I = \begin{pmatrix} B_{(1)}^I & 0 \\ 0 & B_{(2)}^I \end{pmatrix}, \tag{3.2}
$$

where

$$
B_{(1)}^i = \frac{\mu}{3} J_{(1)}^i, \quad B_{(1)}^a = 0 \cdot I_{N_1 \times N_1}, \tag{3.3}
$$

represents a spherical membrane sitting at the origin of all the transverse directions of pp wave, and

$$
B_{(2)}^i = -\frac{\mu}{3} J_{(2)}^i, \quad B_{(2)}^a = x^a(t) \cdot I_{N_2 \times N_2}, \tag{3.4}
$$

represents an antimembrane sitting at the origin of the $1, 2, 3$ directions and moving along the trajectory $x^a(t)$ in the $4, \ldots, 9$ directions, with the extra minus sign in $B_{(2)}^i$ appropriate for an antimembrane as explained earlier. $J_{(s)}^i, s = 1, 2$ is an $N_s \times N_s$ dimensional irreducible representation of $\mathfrak{su}(2)$ with $\left[J_{(s)}^i, J_{(s)}^j\right] = i\epsilon^{ijk} J_{(s)}^k$. The background values for the gauge field A and the fermions all vanish. Recall that the Casimir of the N_s-dimensional irreducible representation of $\mathfrak{su}(2)$ is given by $J_{(s)}^i J_{(s)}^i = \frac{N_s^2 - 1}{4} \cdot I_{N_s \times N_s}$; hence the radius of the membrane is $r_0 = \sqrt{\frac{\mathrm{Tr}\left(B_{(1)}^i B_{(1)}^i\right)}{N_1}} = \frac{\mu}{6}\sqrt{N_1^2 - 1}$ (which is approximately $\frac{\mu}{6} N_1$, for large N_1, or $\frac{\alpha\mu}{6} N_1$ upon restoring α), and that of the antimembrane is $r_0' = \sqrt{\frac{\mathrm{Tr}\left(B_{(2)}^i B_{(2)}^i\right)}{N_2}} = \frac{\mu}{6}\sqrt{N_2^2 - 1}$.

Plugging the above background configuration B^I into (3.1), we find that contributions from $B_{(1)}^I$ cancel out as expected (since a lone spherical membrane is supersymmetric) and what is left comes purely from the antimembrane,

$$
S_{\mathrm{tree}} = \int dt \, N_2 \left\{ \frac{1}{2} \dot{x}^a \dot{x}^a - 2\left(\frac{\mu}{3}\right)^4 \frac{\mathrm{Tr} \, J_{(2)}^k J_{(2)}^k}{N_2} - \frac{1}{2}\left(\frac{\mu}{6}\right)^2 x^a x^a \right\},
$$

$$= \int dt \, N_2 \left\{ \frac{1}{2} \dot{x}^a \dot{x}^a - 2 \left(\frac{\mu}{3} \right)^4 \frac{N_2^2 - 1}{4} - \frac{1}{2} \left(\frac{\mu}{6} \right)^2 x^a x^a \right\}, \tag{3.5}$$

with the subscript "tree" denoting that this is the tree-level action. The second term, i.e. the constant potential term in the above S_{tree}, stems from the fact that the antimembrane with a constant radius r_0' is not supersymmetric and does not satisfy the equation of motion. In fact, recalling that the total M-momentum carried by the antimembrane is N_2, which is equal to $\int \Pi_- = 4\pi \tilde{p}^+$, and also noting that $T = \frac{1}{2\pi}$ (recall $T = \frac{M^3}{2\pi}$), we see that the constant term is equal to the lightcone lagrangian (2.4) in the large N_2 limit (upon setting $r_0' = \frac{\mu \tilde{p}^+}{3T} = \frac{\mu N_2}{6}$, i.e., setting $\eta = 1$). But we've already commented after eq. (2.4) that $\eta = 1$ does not solve the antimembrane equation of motion.[5]

Although an antimembrane with a constant radius does not satisfy the equation of motion, it provides an off-shell background field configuration whose effective potential can be computed. This is just as for ordinary field theories — indeed the BMN matrix theory is just an ordinary quantum mechanical (field) theory. More examples in which off-shell background field configurations in the BMN matrix theory were considered, in order to test gauge-gravity duality, can be found in [14, 15], in which the probe graviton/membrane was allowed to follow an arbitrary off-shell trajectory and complete agreement on the two sides of the duality was found.

3.2 One-loop

The one-loop potential is given by the Coleman-Weinberg formula (also called the "sum-over-mass formula"),

$$V_{\text{eff}}^{\text{one-loop}} = -\frac{1}{2} \left(\sum m_{\text{boson}} - \sum m_{\text{fermion}} - \sum m_{\text{ghost}} \right), \tag{3.6}$$

where the ghosts arise from the standard gauge fixing of the background field method. (For explicit details in the context of the pp wave, see [15, 14, 13].) Upon writing the fluctuations as,

$$X^I = B^I + Y^I, \tag{3.7}$$

and rescaling

$$A \to \mu^{-1/2} A, \quad Y^I \to \mu^{-1/2} Y^I, \quad C \to \mu^{-1/2} C, \quad \bar{C} \to \mu^{-1/2} \bar{C}, \quad B^I \to \mu B^I, \quad t \to \mu^{-1} t, \tag{3.8}$$

the part of the action that is quadratic in the fluctuating fields no longer contains μ explicitly and is given by

$$S_2 = \int dt \, \text{Tr} \left\{ \frac{1}{2} (\dot{Y}^I)^2 - 2i \dot{B}^I [A, Y^I] + \frac{1}{2} ([B^I, Y^J])^2 + [B^I, B^J][Y^I, Y^J] - i \epsilon^{ijk} B^i Y^j Y^k - \right.$$
$$\left. - \frac{1}{2} \left(\frac{1}{3} \right)^2 (Y^i)^2 - \frac{1}{2} \left(\frac{1}{6} \right)^2 (Y^a)^2 + i \Psi^\dagger \dot{\Psi} - \Psi^\dagger \gamma^I [\Psi, B^I] - i \frac{1}{4} \Psi^\dagger \gamma^{123} \Psi - \right.$$

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[5]To get an antimembrane of radius other than $\frac{\mu N_2}{6}$, i.e. with $\eta \neq 1$, one takes $B_{(2)}^i = -h\frac{\mu}{3} J_{(2)}^i$ with h being a pure number which then gives $r_0' = h\frac{\mu N_2}{6}$, i.e. $\eta = h$. Similarly for the membrane.

$$-\frac{1}{2}\left(\dot{A}\right)^2 - \frac{1}{2}\left([B^I, A]\right)^2 + \dot{\bar{C}}\dot{C} + [B^I, \bar{C}][B^I, C]\bigg\}, \qquad (3.9)$$

where the γ^I's are 16×16 real and symmetric SO(9) gamma matrices. Write the flucuating fields in block form

$$A = \begin{pmatrix} Z^0_{(1)} & \Phi^0 \\ \Phi^{0\dagger} & Z^0_{(2)} \end{pmatrix}, \qquad Y^I = \begin{pmatrix} Z^I_{(1)} & \Phi^I \\ \Phi^{I\dagger} & Z^I_{(2)} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \Psi_{(1)} & \chi \\ \chi^{\dagger} & \Psi_{(2)} \end{pmatrix},$$

$$C = \begin{pmatrix} C_{(1)} & C \\ C^{\dagger} & C_{(2)} \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} \bar{C}_{(1)} & \bar{C} \\ \bar{C}^{\dagger} & \bar{C}_{(2)} \end{pmatrix}. \qquad (3.10)$$

It is easy to see that the contributions to S_2 from the bosons, fermions, and ghosts separate,

$$S_2 = S_{\text{boson}} + S_{\text{fermion}} + S_{\text{ghost}}. \qquad (3.11)$$

Furthermore, one can see that the contribution from the diagonal fluctuations $Z^0_{(s)}, Z^I_{(s)}$, etc. (which are $N_s \times N_s$ matrices, $s = 1, 2$) and that from the off-diagonal fluctuations $\Phi^0, \Phi^{0\dagger}$, etc. (which are $N_1 \times N_2$ or $N_2 \times N_1$ matrices) also separate (using the subscript "d" to denote "diagonal" and "o.d." to denote "off-diagonal")

$$S_{\text{boson}} = (S_{\text{boson}})_{\text{d}} + (S_{\text{boson}})_{\text{o.d.}}, \qquad S_{\text{fermion}} = (S_{\text{fermion}})_{\text{d}} + (S_{\text{fermion}})_{\text{o.d.}},$$

$$S_{\text{ghost}} = (S_{\text{ghost}})_{\text{d}} + (S_{\text{ghost}})_{\text{o.d.}} \qquad (3.12)$$

3.2.1 Diagonal fluctuations

Below we shall first look at the mass spectrum of the diagonal fluctuations. Again, one sees that the contributions from the membrane block and that from the antimembrane block separate, i.e.

$$(S_{\text{boson}})_{\text{d}} = \sum_{s=1,2}(S_{\text{boson}})_{\text{d}(s)}, \quad (S_{\text{fermion}})_{\text{d}} = \sum_{s=1,2}(S_{\text{fermion}})_{\text{d}(s)}, \quad (S_{\text{ghost}})_{\text{d}} = \sum_{s=1,2}(S_{\text{ghost}})_{\text{d}(s)}. \qquad (3.13)$$

Recall that the background configuration after rescaling by μ is given by

$$B^i_{(s)} = \eta_s \frac{1}{3} J^i_{(s)}, \qquad s = 1, 2, \qquad (3.14)$$

with $\eta_1 = 1$ and $\eta_2 = -1$. It is straightforward to show that

$$(S_{\text{boson}})_{\text{d}(s)} = \text{Tr}\left\{ \frac{1}{2}(\dot{Z}^I_{(s)})^2 - \frac{1}{2}(\dot{Z}^0_{(s)})^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2 \left[J^i_{(s)}, Z^J_{(s)}\right]^2 + \left(\frac{1}{3}\right)^2 i\epsilon^{ijk}\left[J^k_{(s)}, Z^i_{(s)}\right]Z^j_{(s)} - \right.$$

$$- \frac{1}{2}i\epsilon^{ijk}\eta_s\frac{1}{3}\left[J^i_{(s)}, Z^j_{(s)}\right]Z^k_{(s)} - \left(\frac{1}{3}\right)^2\frac{1}{2}(Z^i_{(s)})^2 -$$

$$\left. - \left(\frac{1}{6}\right)^2\frac{1}{2}(Z^a_{(s)})^2 - \frac{1}{2}\left(\frac{1}{3}\right)^2\left[J^i_{(s)}, Z^0_{(s)}\right]^2 \right\}. \qquad (3.15)$$

Using the fact

$$
\mathrm{Tr}\left\{\left(\frac{1}{3}\right)^2 \epsilon^{ijk}\epsilon^{ilm}\left[J^j_{(s)}, Z^k_{(s)}\right]\left[J^l_{(s)}, Z^m_{(s)}\right]\right\} =
$$
$$
= \mathrm{Tr}\left\{\left(\frac{1}{3}\right)^2 \left(\left[J^j_{(s)}, Z^k_{(s)}\right]^2 - \left[J^i_{(s)}, Z^i_{(s)}\right]^2 + i\epsilon^{ijk}Z^i_{(s)}\left[J^j_{(s)}, Z^k_{(s)}\right]\right)\right\}, \quad (3.16)
$$

one can rewrite (3.15) as

$$
(S_{\mathrm{boson}})_{\mathrm{d}(s)} = \frac{1}{2}\,\mathrm{Tr}\left\{ -(\dot{Z}^0_{(s)})^2 - \left(\frac{1}{3}\right)^2\left[J^i_{(s)}, Z^0_{(s)}\right]^2 + (\dot{Z}^i_{(s)})^2 - \right.
$$
$$
- \left(\frac{1}{3}\right)^2\left(Z^i_{(s)} + i\epsilon^{ijk}\eta_s\left[J^j_{(s)}, Z^k_{(s)}\right]\right)^2 +
$$
$$
+ \left(\frac{1}{3}\right)^2\left[J^i_{(s)}, Z^i_{(s)}\right]^2 + (\dot{Z}^a_{(s)})^2 - \left(\frac{1}{3}\right)^2\left(\frac{1}{4}(Z^a_{(s)})^2 - \left[J^i_{(s)}, Z^a_{(s)}\right]^2\right) +
$$
$$
\left. + (1-\eta_s)\left(\frac{1}{3}\right)^2 i\epsilon^{ijk}Z^i_{(s)}\left[J^j_{(s)}, Z^k_{(s)}\right]\right\}. \quad (3.17)
$$

All but the last line of eq. (3.17) can be obtained by replacing $J^i_{(s)}$ in equation (5.2) of [12] with $\eta_s J^i_{(s)}$. For $\eta_s = 1$, the last line vanishes, of course, and one finds that the membrane configuration satisfies the equations of motion with zero energy — i.e. it is BPS. For $\eta_s = -1$, the last line explicitly demonstrates the absence of supersymmetry for the antimembrane configuration.

Now we want to diagonalize $(S_{\mathrm{boson}})_{\mathrm{d}(s)}$. We use the $N_s \times N_s$ matrix spherical harmonics $Y^{(s)}_{jm}$ ($j = 0,\ldots,N_s - 1$; $m = -j,\ldots,j$) to expand the fields, e.g.

$$
Z^0_{(s)} = \sum_{j=0}^{N_s-1}\sum_{m=-j}^{j} Z^0_{(s)jm} Y^{(s)}_{jm}. \quad (3.18)
$$

As $Z^0_{(s)}$ is not coupled to other fields, finding its mass is trivial. By noticing that

$$
\mathrm{Tr}(\dot{Z}^0_{(s)})^2 = \sum_{j=0}^{N_s-1}\sum_{m=-j}^{j} N_s\left|\dot{Z}^0_{(s)jm}\right|^2,
$$
$$
\mathrm{Tr}\left(\left[J^i_{(s)}, Z^0_{(s)}\right]\right)^2 = -\sum_{j=0}^{N_s-1}\sum_{m=-j}^{j} j(j+1)N_s\left|Z^0_{(s)jm}\right|^2, \quad (3.19)
$$

one finds the mass of $Z^0_{(s)jm}$ to be

$$
\frac{1}{3}\sqrt{j(j+1)}. \quad (3.20)
$$

Similarly $Z^a_{(s)}$ is not coupled to other fields and one finds the mass of $Z^a_{(s)jm}$ to be

$$
\frac{1}{3}\sqrt{\frac{1}{4} + j(j+1)} = \frac{1}{3}\left(j + \frac{1}{2}\right). \quad (3.21)
$$

The $Z^i_{(s)}$'s are coupled and finding their masses requires slightly more work. We relegate the details to appendix A, where the masses and degeneracies for $s = 1$ are given in eq. (A.5), (A.1), while the masses and degeneracies for $s = 2$ are given in eqs. (A.6) and (A.1). This is summarized in table 1(a).

It is worth pointing out that there are twenty one tachyon modes in the antimembrane spectrum (A.6), corresponding to mass $\frac{1}{3}\sqrt{j^2 - 4j + 1}$ for $j = 1, 2, 3$, with the masses-squared being $-2\left(\frac{1}{3}\right)^2$, $-3\left(\frac{1}{3}\right)^2$, $-2\left(\frac{1}{3}\right)^2$, and the degeneracies being $2j + 3 = 5, 7, 9$ respectively. These correspond to instabilities in the fluctuations of $X^{1,2,3}$ and will cause the antimembrane to decay.

Next we look at the contribution from diagonal fluctuations of the fermion. This part of the action (3.9) is given by

$$(S_{\text{fermion}})_{\text{d}(s)} = \text{Tr}\left(i\Psi^\dagger_{(s)}\dot{\Psi}_{(s)} - \frac{\eta_s}{3}\Psi^\dagger_{(s)}\gamma^i[\Psi_{(s)}, J^i_{(s)}] - i\frac{1}{4}\Psi^\dagger_{(s)}\gamma^{123}\Psi_{(s)}\right),$$

$$= \text{Tr}\left(i\psi^{\dagger A\alpha}_{(s)}\dot{\psi}_{(s)A\alpha} + \frac{\eta_s}{3}\psi^{\dagger A\alpha}_{(s)}(\sigma^i)_\alpha{}^\beta[\psi_{(s)A\beta}, J^i_{(s)}] - \frac{1}{4}\psi^{\dagger A\alpha}_{(s)}\psi_{(s)A\alpha}\right), \quad (3.22)$$

where in the last line we have written the spinor in the SU(2)×SU(4) form $\psi_{A\alpha}$ with α being SU(2) index and A being SU(4) index, and σ^i is the standard Pauli matrix. One immediately sees that for solutions of the eigenvalue problem

$$(\sigma^i)_\alpha{}^\beta\left[J^i_{(s)}, \psi_{(s)A\beta}\right] = \lambda\psi_{(s)A\alpha}, \quad (3.23)$$

the action is diagonalized with the mass of $\psi_{(s)A\alpha}$ given by $\left|\frac{\eta_s}{3}\lambda + \frac{1}{4}\right|$. This eigenvalue problem is solved by the matrix spinor spherical harmonics [22] (see also [23, 12]) so we just quote the result,

$$
\begin{aligned}
\lambda &= j, &&\text{with } j = 0, \ldots, N_s - 1, && m = -j - 1, \ldots, j, \\
\lambda &= -j - 1, &&\text{with } j = 1, \ldots, N_s - 1, && m = -j, \ldots, j - 1, \quad (3.24)
\end{aligned}
$$

which gives the masses summarized in table 1(b).

As for the ghost part of the action, there is no difference between $s = 1$ and $s = 2$, and for both $\eta_s = \pm 1$ the masses of the ghosts are $\frac{1}{3}\sqrt{j(j+1)}$ with $j = 0, \ldots, N_s - 1$ and $m = -j, \ldots, j$. This completes our presentation of the mass spectrum of the diagonal fluctuations; see table 1(b). Note that this spectrum is independent of the antimembrane's motion $x^a(t)$ in the $x^4, \ldots, x^9$ directions. As a result, $x^a(t)$ (and its time-derivative) does not appear in the part of the one-loop effective potential coming from the diagonal fluctuations, which we turn to below.

3.2.2 One-loop effective potential for a single antimembrane

Using the mass spectrum of the diagonal fluctuations found in section 3.2.1 we can now compute the one-loop potential for the membrane alone and also for the antimembrane alone — i.e. that part of the potential that does not include the interaction between the two. For the membrane, the mass spectrum we gave above is the same as that in [12], for which $V^{\text{one-loop}}_{\text{eff}}$ vanishes as expected (since it is supersymmetric). On the other hand,

	mass	range of j	degeneracy
$\eta_s = 1$	$\frac{1}{3}j$	$1\leq j \leq N_s - 1$	$2j - 1$
	$\frac{1}{3}\sqrt{j(j+1)}$	$1\leq j \leq N_s - 1$	$2j + 1$
	$\frac{1}{3}(j+1)$	$0\leq j \leq N_s - 1$	$2j + 3$
$\eta_s = -1$	$\frac{1}{3}\sqrt{j^2 + 6j + 6}$	$1\leq j \leq N_s - 1$	$2j - 1$
	$\frac{1}{3}\sqrt{j^2 + j + 6}$	$1\leq j \leq N_s - 1$	$2j + 1$
	$\frac{1}{3}\sqrt{j^2 - 4j + 1}$	$0\leq j \leq N_s - 1$	$2j + 3$

(a) Z^i

	mass	range of j	degeneracy
$\eta_s = 1$	$\frac{1}{3}\left(j + \frac{3}{4}\right)$	$0\leq j \leq N_s - 1$	$2j + 2$
	$\frac{1}{3}\left(j + \frac{1}{4}\right)$	$1\leq j \leq N_s - 1$	$2j$
$\eta_s = -1$	$\left|\frac{j}{3} - \frac{1}{4}\right|$	$0\leq j \leq N_s - 1$	$2j + 2$
	$\frac{j}{3} + \frac{7}{12}$	$1\leq j \leq N_s - 1$	$2j$

(b) fermions

mass	range of j	degeneracy
$\frac{1}{3}\sqrt{j(j+1)}$	$0\leq j \leq N_s - 1$	$2j + 1$

(c) Z^0

mass	range of j	degeneracy
$\frac{1}{3}\left(j + \frac{1}{2}\right)$	$0\leq j \leq N_s - 1$	$2j + 1$

(d) Z^a

mass	range of j	degeneracy
$\frac{1}{3}\sqrt{j(j+1)}$	$0\leq j \leq N_s - 1$	$2j + 1$

(e) ghosts

Table 1: The mass spectrum of the diagonal blocks.

for the antimembrane all the supersymmetries are broken and the one-loop potential does not vanish. Before writing down the formula for the effective potential, we first undo the rescaling done on the fields and time in eq. (3.8) by replacing $v \to \frac{v}{\mu}$, $V_{\text{eff}} \to \mu V_{\text{eff}}$, and also restore powers of α. One then finds

$$
\begin{aligned}
\bar{V}_{\text{eff}}^{\text{one-loop}} = -\frac{1}{2}\mu \Bigg[& \sum_{j=0}^{N_2-1}(2j+1)\frac{1}{3}\sqrt{j(j+1)} + 6\sum_{j=0}^{N_2-1}(2j+1)\frac{1}{3}\left(j + \frac{1}{2}\right) + \\
& + \sum_{j=1}^{N_2-1}(2j-1)\frac{1}{3}\sqrt{j^2 + 6j + 6} + \sum_{j=1}^{N_2-1}(2j+1)\frac{1}{3}\sqrt{j^2 + j + 6} + \\
& + \sum_{j=0}^{N_2-1}(2j+3)\frac{1}{3}\sqrt{j^2 - 4j + 1} - 4\sum_{j=0}^{N_2-1}(2j+2)\left|\frac{j}{3} - \frac{1}{4}\right| - \\
& - 4\sum_{j=1}^{N_2-1}2j\left(\frac{j}{3} + \frac{7}{12}\right) - 2\sum_{j=0}^{N_2-1}(2j+1)\frac{1}{3}\sqrt{j(j+1)} \Bigg].
\end{aligned}
\tag{3.25}
$$

To expand the above summation (3.25) for large N_2, we use the Euler-Maclaurin

formula,

$$\sum_{k=1}^{n-1} f(k) = \int_0^n f(k)dk - \frac{1}{2}[f(n) + f(0)] + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(n) - f^{(2m-1)}(0)\right], \quad (3.26)$$

where $f^{(m)}(k)$ stands for the mth-derivative of f, and B_{2m} is the Bernoulli number ($\frac{B_2}{2!} = \frac{1}{12}$, $\frac{B_4}{4!} = -\frac{1}{720}$, $\frac{B_6}{6!} = \frac{1}{30240}$, $\frac{B_8}{8!} = -\frac{1}{1209600}$, and so on). Applying this to the summation (3.25) (using Mathematica®), we find that all terms with positive powers of N_2 cancel, leaving

$$\bar{V}_{\text{eff}}^{\text{one-loop}}(r_0') = -i\left[\frac{7\mu}{6}(2\sqrt{2} + \sqrt{3})\right] + \left\{-\frac{\mu}{2}\left[b + \frac{87}{4}\frac{1}{N_2} + \mathcal{O}\left(\frac{1}{N_2^2}\right)\right]\right\}. \quad (3.27)$$

Note that $\bar{V}_{\text{eff}}^{\text{one-loop}}$ has a constant imaginary part coming from the twenty-one tachyon modes in the antimembrane's fluctuations along the $X^{1,2,3}$ directions, which gives a constant decay rate. The pure number b in the real part of $\bar{V}_{\text{eff}}^{\text{one-loop}}$ has an approximate value of -4.56 and is just the zero-point energy of the nonsupersymmetric antimembrane.

3.2.3 Off-diagonal fluctuations

For any $N_r \times N_s$ matrix M define

$$\{J^i, M\} \equiv J_{(r)}^i M + M J_{(s)}^i, \quad (3.28)$$

and

$$[J^i, M] \equiv J_{(r)}^i M - M J_{(s)}^i. \quad (3.29)$$

(When $N_r = N_s$, this notation goes without saying, of course.) One useful identity is

$$\{J^i, \{J^i, M\}\} = 2(\Lambda_{(r)} + \Lambda_{(s)})M - [J^i, [J^i, M]]. \quad (3.30)$$

with $\Lambda_{(r)} = \frac{N_r^2 - 1}{4}$ being the Casimir $J_{(r)}^i J_{(r)}^i = \Lambda_{(r)} \cdot I_{N_r \times N_r}$.

The off-diagonal fluctuations give the interaction between the membrane and the antimembrane. Let us first look at the bosonic part. As can be readily seen, the Φ^0, Φ^a part of the action and the $\Phi^{1,2,3}$ part are not coupled, i.e., $(S_{\text{boson}})_{\text{o.d.}} = (S_{\text{boson}})_{\text{o.d.},0a} + (S_{\text{boson}})_{\text{o.d.},123}$, with

$$(S_{\text{boson}})_{\text{o.d.},0a} = \text{Tr}\left(\dot{\Phi}^a \dot{\Phi}^{a\dagger} - 2i\frac{v^a}{\mu}(\Phi^{0\dagger}\Phi^a - \Phi^{a\dagger}\Phi^0) - \left(\frac{1}{3}\right)^2 [J^i, \Phi^a][J^i, \Phi^{a\dagger}] - \right.$$
$$- \dot{\Phi}^0 \dot{\Phi}^{0\dagger} + \left(\frac{1}{3}\right)^2 [J^i, \Phi^0][J^i, \Phi^{0\dagger}] +$$
$$+ \left[2\left(\frac{1}{3}\right)^2 (\Lambda_1 + \Lambda_{(2)}) + \frac{x^b x^b}{\mu^2}\right]\Phi^0 \Phi^{0\dagger} -$$
$$\left. - \left[2\left(\frac{1}{3}\right)^2 (\Lambda_1 + \Lambda_{(2)}) + \left(\frac{1}{6}\right)^2 + \frac{x^b x^b}{\mu^2}\right]\Phi^a \Phi^{a\dagger}\right), \quad (3.31)$$

– 11 –

(with $v^a \equiv \dot{x}^a$) and

$$(S_{\text{boson}})_{\text{o.d.},123} = \text{Tr} \left(\dot{\Phi}^i \dot{\Phi}^{i\dagger} - \left(\frac{1}{3}\right)^2 \{J^i, \Phi^j\} \{J^i, \Phi^{j\dagger}\} - \frac{x^b x^b}{\mu^2} \Phi^i \Phi^{i\dagger} + \right. \tag{3.32}$$

$$\left. + 2\left(\frac{1}{3}\right)^2 i\epsilon^{ijk} \left[J^k, \Phi^i\right] \Phi^{j\dagger} - i\epsilon^{ijk} \left(\frac{1}{3}\right) \{J^i, \Phi^j\} \Phi^{k\dagger} - \left(\frac{1}{3}\right)^2 \Phi^i \Phi^{i\dagger} \right).$$

The masses of Φ^0, Φ^a can be readily found. Making use of the SO(6) symmetry to set $v^a = v\delta^{a9}$, we see that Φ^0, Φ^9 are coupled, while $\Phi^{4,5,6,7,8}$ are decoupled. One then expands the fields using the $N_1 \times N_2$ matrix spherical harmonics

$$\Phi^I = \sum_{j=\frac{|N_1-N_2|}{2}}^{\frac{N_1+N_2}{2}-1} \sum_{m=-j}^{j} \Phi^I_{jm} Y_{jm}. \tag{3.33}$$

The masses of $\Phi^{4,5,6,7,8}$ are then immediately seen to be

$$\sqrt{2\left(\frac{1}{3}\right)^2 (\Lambda_{(1)} + \Lambda_{(2)}) + \left(\frac{1}{6}\right)^2 + \frac{x^b x^b}{\mu^2} - \left(\frac{1}{3}\right)^2 j(j+1)}, \tag{3.34}$$

with $j = \frac{|N_1-N_2|}{2}, \ldots, \frac{N_1+N_2}{2} - 1$ and $m = -j, \ldots, j$. For Φ^0, Φ^9, we first Wick rotate $A \to iA$ (which means $\Phi^0 \to i\Phi^0$ and $\Phi^{0\dagger} \to i\Phi^{0\dagger}$) and $v \to -iv$. Then one finds that the mass squared matrix for Φ^0, Φ^9 is given by

$$\begin{pmatrix} \tilde{\Delta} & i\frac{2v}{\mu} \\ -i\frac{2v}{\mu} & \tilde{\Delta} + \left(\frac{1}{6}\right)^2 \end{pmatrix}, \tag{3.35}$$

where we have expanded using matrix spherical harmonics and suppressed the jm subscripts (since different jm components do not mix), and $\tilde{\Delta} \equiv 2\left(\frac{1}{3}\right)^2 (\Lambda_{(1)} + \Lambda_{(2)}) + \frac{x^b x^b}{\mu^2} - \left(\frac{1}{3}\right)^2 j(j+1)$. Diagonalizing this 2×2 matrix we get the mass of Φ^0, Φ^9

$$\left[\tilde{\Delta} + \frac{1}{72} \pm \frac{1}{72}\sqrt{1 + \left(\frac{144v}{\mu}\right)^2}\right]^{1/2}, \tag{3.36}$$

with the range of (jm) being $j = \frac{|N_1-N_2|}{2}, \ldots, \frac{N_1+N_2}{2} - 1$ and $m = -j, \ldots, j$.

Finding the masses of $\Phi^{1,2,3}$ involves substantial work and the details are given in appendix B. Those masses contain $x^a(t)$ but not its time-derivative v.

The mass spectrum of the off-diagonal fermionic fluctuation χ is computed in appendix C and summarized in eqs. (C.24) and (C.25).

The off-diagonal ghost part of the action is given by

$$(S_{\text{ghost}})_{\text{o.d.}} = \text{Tr} \left((\dot{\bar{C}}\dot{C}^\dagger + \dot{\bar{C}}^\dagger \dot{C}) - \left(\frac{1}{3}\right)^2 \{J^i, \bar{C}\} \{J^i, C^\dagger\} - \frac{x^a x^a}{\mu^2} \bar{C} C^\dagger - \right.$$

$$-\left(\frac{1}{3}\right)^2 \left\{J^i, \bar{C}^\dagger\right\} \left\{J^i, C\right\} - \frac{x^a x^a}{\mu^2} \bar{C}^\dagger C \Bigg). \tag{3.37}$$

Using

$$\text{Tr}\left(-\left\{J^i, \bar{C}\right\}\left\{J^i, C^\dagger\right\}\right) = \text{Tr}\left(C^\dagger\left\{J^i, \left\{J^i, \bar{C}\right\}\right\}\right)$$
$$= \text{Tr}\left(-2(\Lambda_{(1)} + \Lambda_{(2)})\bar{C}C^\dagger - \left[J^i, \bar{C}\right]\left[J^i, C^\dagger\right]\right), \tag{3.38}$$

one readily finds that the masses of $\bar{C}$ and $C^\dagger$ (with their complex conjugates being $\bar{C}^\dagger$ and C respectively) are the same and are given by

$$\sqrt{2\left(\frac{1}{3}\right)^2 (\Lambda_{(1)} + \Lambda_{(2)}) + \frac{x^a x^a}{\mu^2} - \left(\frac{1}{3}\right)^2 j(j+1)}, \tag{3.39}$$

with $j = \frac{|N_1 - N_2|}{2}, \ldots, \frac{N_1 + N_2}{2} - 1$ and $m = -j, \ldots, j$.

3.2.4 Interaction between the membrane and the antimembrane

The part of the one-loop potential describing the interaction between the membrane block and the antimembrane block is obtained using the mass spectrum of the off-diagonal fluctuations worked out in section 3.2.3. This is done by writing down the sum-over-mass expression (3.6), using the Euler-Maclaurin formula (3.26), replacing N_2 by $\frac{6r_0'}{\alpha\mu}$, N_1 by $\frac{6r_0}{\alpha\mu}$, then expanding in powers of α and in the end dropping terms at quadratic or higher orders in α. The reason for this α-expansion is the same as explained in [15]: as we shall soon see in section 4, the interaction on the supergravity side is $\mathcal{O}\left(\alpha^1\right)$; since here we are only interested in comparing matrix theory predictions to supergravity results, on the matrix theory side we can also just keep order α^1. Indeed, higher orders of α come with higher powers of $\frac{1}{r}$, thus becoming important only at short distances; these are matrix theory corrections beyond supergravity.[6]

We shall compute the velocity-dependent part of the effective action, $V_{\text{eff},v\text{-dep.}}^{\text{one-loop}}$, which can be seen to receive contributions only from the masses of Φ^0, Φ^9 and the fermion χ. After restoring powers of α and μ as described above eq. (3.25), the sum-over-mass for Φ^0, Φ^9, χ is (assuming that $N_2 > N_1$, i.e. that the difference between the radii of the antimembrane and the membrane $w \equiv r_0' - r_0 > 0$, and defining $z \equiv \sqrt{x^a x^a}$ to be the separation of the two objects in the $x^4, \ldots, x^9$ directions)

$$V_{\text{eff},v\text{-dep.}}^{\text{one-loop}} =$$
$$= -\frac{1}{2\alpha}\left\{2 \sum_{j=\frac{|N_1 - N_2|}{2}}^{\frac{N_1+N_2}{2}-1} (2j+1) \times \right.$$

---

[6]In section 3.2.2, we did a $1/N_2$ expansion to compute the effective potential for a single antimembrane. Since the only α-dependence appears via $N_2 = \frac{6r_0'}{\alpha\mu}$, this amounted to an α-expansion. Of course it has to be this way; despite the fact that diagonal fluctuations were considered there and off-diagonal fluctuations here, they both belong to the same quantity, namely the one-loop effective potential.

$$\times \left[\sqrt{z^2 + \left(\frac{\alpha\mu}{3}\right)^2 \left[\frac{N_1^2 + N_2^2 - 2}{2} - j(j+1)\right] + \frac{(\alpha\mu)^2}{72} + \frac{1}{72}\sqrt{(\alpha\mu)^4 + (144\alpha v)^2}} + \right.$$

$$+ \left. \sqrt{z^2 + \left(\frac{\alpha\mu}{3}\right)^2 \left[\frac{N_1^2 + N_2^2 - 2}{2} - j(j+1)\right] + \frac{(\alpha\mu)^2}{72} - \frac{1}{72}\sqrt{(\alpha\mu)^4 + (144\alpha v)^2}} \right] -$$

$$- 4(N_1 + N_2)\sqrt{z^2 + (\alpha\mu)^2 \left(\frac{N_2 - N_1}{6} - \frac{1}{4}\right)^2 + \alpha v} -$$

$$- 4(N_2 - N_1)\sqrt{z^2 + (\alpha\mu)^2 \left(-\frac{N_1 + N_2}{6} - \frac{1}{4}\right)^2 + \alpha v} -$$

$$- 4 \sum_{l=\frac{|N_1 - N_2|}{2}}^{\frac{N_1 + N_2}{2} - 2} 2(l+1)\left[\sqrt{z^2 + (\alpha\mu)^2 \left(\frac{1}{3}\sqrt{\frac{N_1^2 + N_2^2}{2} - (l+1)^2} - \frac{1}{4}\right)^2 + \alpha v} + \right.$$

$$+ \left. \sqrt{z^2 + (\alpha\mu)^2 \left(-\frac{1}{3}\sqrt{\frac{N_1^2 + N_2^2}{2} - (l+1)^2} - \frac{1}{4}\right)^2 + \alpha v} \right] -$$

$$- 4(N_1 + N_2)\sqrt{z^2 + (\alpha\mu)^2 \left(\frac{N_2 - N_1}{6} - \frac{1}{4}\right)^2 - \alpha v} -$$

$$- 4(N_2 - N_1)\sqrt{z^2 + (\alpha\mu)^2 \left(-\frac{N_1 + N_2}{6} - \frac{1}{4}\right)^2 - \alpha v} -$$

$$- 4 \sum_{l=\frac{|N_1 - N_2|}{2}}^{\frac{N_1 + N_2}{2} - 2} 2(l+1)\left[\sqrt{z^2 + (\alpha\mu)^2 \left(\frac{1}{3}\sqrt{\frac{N_1^2 + N_2^2}{2} - (l+1)^2} - \frac{1}{4}\right)^2 - \alpha v} + \right. \quad (3.40)$$

$$\left. + \sqrt{z^2 + (\alpha\mu)^2 \left(-\frac{1}{3}\sqrt{\frac{N_1^2 + N_2^2}{2} - (l+1)^2} - \frac{1}{4}\right)^2 - \alpha v} \right] \right\}.$$

Carrying out the procedures outlined above, we find that, for the velocity-dependent part of the above $V_{\text{eff},v\text{-dep.}}^{\text{one-loop}}$, terms at $O(\alpha^0)$ as well as terms with negative powers of α all cancel out, and and the final result is (after Wick rotating back, i.e. $v \to iv$)

$$V_{\text{eff},v\text{-dep.}}^{\text{one-loop}} = \frac{9\alpha}{\mu^2(w^2 + z^2)^{5/2}} \left\{\frac{3}{8}v^4 + v^2 r_0^2 \mu^2 \left[\frac{1}{3} + \frac{5}{6}\frac{w}{r_0} + \frac{1}{48}\frac{26w^2 + z^2}{r_0^2}\right]\right\}. \quad (3.41)$$

The final step in getting the result (3.41) is an additional expansion called the "near-membrane expansion" in which we expand in the parameter $\frac{\xi}{r_0}$ (where $\xi \equiv \sqrt{w^2 + z^2}$) and then only keep the leading terms (which are the terms that diverge as $\xi \to 0$). The only reason for doing this near-membrane expansion is that on the supergravity side we have only worked to leading orders of this expansion (see section 4 below and [15]) and we want to compare the result here with the result there. Without doing this expansion, we would obtain the analog of the interpolating potential given in section 6 of [15], which would be valid regardless of whether the separation ξ between the membrane and the anti-membrane is much smaller than their radii or not. Additionally, we have set v to be of order $\mu\xi$, in accordance with section 4 below.

4. Interaction calculated on the supergravity side

On the linearized supergravity side we use the source-probe analysis, treating the membrane as the source and the antimembrane as the probe.[7] The metric and gauge field perturbations $h_{\mu\nu}, a_{\mu\nu\rho}$ produced by the source membrane were computed in [15]. As pointed out in section 2, to get the lightcone lagrangian $\bar{L}_{\text{l.c.}}$ for the antimembrane we just have to take the membrane $L_{\text{l.c.}}$ given in [15] and flip the sign of the terms containing $a_{\mu\nu\rho}$. We shall make the assumption that v is of order $\mu\xi, \xi \equiv \sqrt{w^2 + z^2}$. This is because on-shell trajectories of the antimembrane are in general elliptical orbits in the $x^{4,\ldots,9}$ directions and have $v \sim \mu z \sim \mu\xi$. Although here we do not require the trajectory to be on-shell — the orbit can be of arbitrary shape — we choose it not to be "too off-shell" by requiring its velocity to be of the same order of magnitude as those of on-shell trajectories. Finally, as in [15], since we are only interested in the part of the membrane/antimembrane interaction that diverges as the two objects get closer and closer, in the expression for $\bar{L}_{\text{l.c.}}$ we shall only keep the terms that are singular as $\xi \to 0$ which are the leading terms in the near-membrane expansion.

The result, upon writing $\bar{L}_{\text{l.c.}}$ as the sum of a velocity-dependent part $\bar{L}_{\text{l.c., }v\text{-dep.}}$ and a velocity-independent part $\bar{L}_{\text{l.c., }v\text{-indep.}}$, is

$$\bar{L}_{\text{l.c., }v\text{-dep.}} = \left(\int d\theta d\phi \Pi_- \Delta\right) \frac{1}{(w^2 + z^2)^{5/2}} \left\{\frac{3}{8}v^4 + v^2 r_0^2 \mu^2 \left[\frac{1}{3} + \frac{1}{2}\frac{w}{r_0} + \frac{1}{48}\frac{2w^2 + z^2}{r_0^2}\right]\right\},$$

(4.1)

$$\bar{L}_{\text{lc, }v\text{-indep.}} = \left(\int d\theta d\phi \Pi_- \Delta\right) \frac{\mu^4 r_0^4}{9(w^2 + z^2)^{5/2}} \left\{\frac{2}{3} + \frac{8}{3}\frac{w}{r_0} + \frac{(32w^2 - 11z^2)}{12r_0^2} + \frac{w(8w^2 - 5z^2)}{8r_0^3} + \right.$$
$$\left. + \frac{(128w^4 + 252w^2 z^2 + 125z^4)}{384 r_0^4}\right\},$$

(4.2)

where the quantity Δ, which is a proportionality constant in $h_{\mu\nu}, a_{\mu\nu\rho}$, is given by $\frac{\kappa_{11}^2 T}{16\pi^4 R\left(\frac{\mu r_0}{3}\right)}$ (see eq. (76) of [15]). Now recalling that $r_0' = \frac{\mu \tilde{p}^+}{3T} = \frac{\mu \Pi_-}{3T \sin\theta}$ gives $\Pi_- = \frac{3r_0' T \sin\theta}{\mu} = \frac{3r_0 T \sin\theta}{\mu}\left(1 + \frac{w}{r_0}\right)$, $\kappa_{11}^2 = \frac{16\pi^5}{M^9}$, $T = \frac{M^3}{2\pi}$, and $\frac{1}{M^3 R} = \alpha$, we find

$$\int d\theta \, d\phi \, \Pi_- \Delta = \frac{9\alpha}{\mu^2}\left(1 + \frac{w}{r_0}\right).$$

(4.3)

Plugging in this value of $\int d\theta d\phi \Pi_- \Delta$ yields

$$\bar{L}_{\text{l.c.,}v\text{-dep.}} = \frac{9\alpha}{\mu^2 (w^2 + z^2)^{5/2}} \left\{\frac{3}{8}v^4 + v^2 r_0^2 \mu^2 \left[\frac{1}{3} + \frac{5}{6}\frac{w}{r_0} + \frac{26w^2 + z^2}{48r_0^2}\right]\right\},$$

(4.4)

$$\bar{L}_{\text{lc,}v\text{-indep.}} = \frac{\alpha \mu^2 r_0^4}{(w^2 + z^2)^{5/2}} \left\{\frac{2}{3} + \frac{10}{3}\frac{w}{r_0} + \frac{(64w^2 - 11z^2)}{12r_0^2} + \right.$$

---

[7]Since the antimembrane is treated as a probe it does not contribute to the stress-energy tensor, and thus integrability of Einstein equation does not require its trajectory to satisfy equation of motion. See also [14, 15].

$$+ \frac{w(88w^2 - 37z^2)}{24r_0^3} + \frac{(512w^4 + 12w^2z^2 + 125z^4)}{384r_0^4} \Bigg\}, \quad (4.5)$$

where in the final expressions of the $\bar{L}$'s we have again only kept terms that are singular as $\xi \to 0$.

Comparing the matrix theory result (3.41) with the supergravity result (4.4), we see that they completely agree. That is to say, at leading order of large r_0 (i.e. in the flat space limit, which is given by $r_0 \to \infty$, $\mu \to 0$, holding μr_0 fixed) they both reduce to

$$\frac{9\alpha}{\mu^2(w^2 + z^2)^{5/2}} \left(\frac{3}{8}v^4 + \frac{1}{3}v^2r_0^2\mu^2 \right). \quad (4.6)$$

This reproduces the flat space agreement; furthermore, the pp wave corrections to the potentials, i.e. the $\left(\frac{5}{6}\frac{w}{r_0} + \frac{26w^2 + z^2}{48r_0^2} \right)$ terms, also agree. Thus, the flat space agreement is extended to the pp wave. This is quite remarkable.

Let us also compare the membrane-antimembrane interaction found above with interaction of other objects in the pp wave. The velocity-dependent part of the membrane-membrane interaction is (eq. (95) of [15])

$$L_{\text{membrane-membrane}} = \frac{9\alpha}{\mu^2(w^2 + z^2)^{5/2}} \left\{ \frac{3}{8}v^4 - v^2r_0^2\mu^2 \left[\frac{2w^2 + 5z^2}{144r_0^2} \right] \right\}. \quad (4.7)$$

Although in the $X^4, \ldots, X^9$ directions the membrane-antimembrane pair are point particles like gravitons, the interaction in these directions is not the same as that of gravitons. In fact, the ratio of the coefficient of the $v^2\mu^2z^2$ term to that of the v^4 term for the membrane-antimembrane interaction (4.4) is $\frac{1}{18}$, while for the graviton-graviton interaction this ratio is different, being given by $\frac{7}{90}$ (see equation (97) of [15]).

5. Discussion

In this work, we examined a spherical membrane and antimembrane in the M-theory pp wave using both M(atrix) theory, and supergravity. We have seen that the one-loop potential reproduces the interactions seen in the supergravity from a probe anlaysis. This remarkable agreement, for a nonsupersymmetric system, does not just provide more evidence for matrix theory. It also leads to additional interesting questions as, given the lack of supersymmetry in the system, one might have expected the potential to be renormalized towards (naïve) disagreement. As mentioned in the Introduction, it is possible that the membrane-antimembrane pair in pp wave is *approximately* supersymmetric in a way similar to a membrane-antimembrane pair in flat space. That would explain the agreement we have found.

We have put more emphasis on the interaction between the membrane and antimembrane as a way of testing the gauge/gravity duality in a nonsupersymmetric setting, and only briefly talked about of the dynamics of a single antimembrane in the pp wave. However, it is worth pointing out the antimembrane by itself deserves further investigation. First, we have found that its fluctuations have tachyon modes and hence one would like

to better understand what is the final product of the corresponding decay process. One natural guess is these tachyon modes cause the antimembrane to deform (and perhaps finally disintegrate), during which gravitational as well as three-form gauge field radiation is emitted. Alternatively, one might suppose that the antimembrane collapses and passes through itself, thereby inverting its dipole to become a membrane, cf. [24], perhaps emitting radiation in the process. (In fact, the first guess is a special case of the second, in which the final membrane state is in the trivial SU(2) vacuum.) It would be interesting to show what happens by a concrete computation. Secondly, we have put the antimembrane off-shell at a constant radius $r_0' = \frac{\mu \tilde{p}^+}{3T}$. By looking at the antimembrane's lagrangian (2.4) we see that the trajectory solving the classical e.o.m., which depicts a collapsing antimembrane, is given by elliptic integrals. The recent work [25] considered spherical D-branes in flat space which collapse due to its tension. The trajectory there possesses a large-small duality relating r to $1/r$, which comes from complex multiplication properties of the Jacobi elliptic functions. It is therefore not inconceivable that our antimembrane will also exhibit this large-small duality, and it would be interesting to work out the details and try to understand the physical implications. This duality, as observed in [25], is probably a disguise of T-duality. A better understanding of that might shed some light on the issue of duality transformations for spherical branes discussed in the Introduction. This, in turn, could help explain our precise agreement between the matrix theory and the supergravity.

We have left the matrix theory computation of the velocity-independent part of the one-loop interaction between the membrane and the antimembrane to future work (the interaction on the supergravity side is given in eq. (4.5)). The necessary ingredients for that computation (i.e., the masses of $\Phi^{1,2,3}$) are worked out in appendix B, and are fairly complicated expressions.

Acknowledgments

A. Finding the masses of $Z_{(s)}^i$

We would like to diagonalize the $Z_{(s)}^i$ part of the action (3.17). This is easily done using the vector spherical harmonics Y_{jlm}^i [22] (see also [23]), for which

$$
\begin{aligned}
\epsilon_{ijk}\left[J^j, Y_{j-1,jm}^k\right] &= i(j+1)Y_{j-1,jm}^i, & j&=1,\ldots,N_s-1; \, m=-j+1,\ldots,j-1,\\
\epsilon_{ijk}\left[J^j, Y_{jjm}^k\right] &= iY_{jjm}^i, & j&=1,\ldots,N_s-1; \, m=-j,\ldots,j,\\
\epsilon_{ijk}\left[J^j, Y_{j+1,jm}^k\right] &= -ijY_{j+1,jm}^i, & j&=0,\ldots,N_s-1; \, m=-j-1,\ldots,j+1. \quad \text{(A.1)}
\end{aligned}
$$

and

$$\left[J^i, Y^i_{jlm}\right] = \sqrt{j(j+1)}\delta_{jl}Y_{lm}. \tag{A.2}$$

One can check that the total number of vector spherical harmonics is $3N_s^2$, as it should be.

Thus one can expand

$$Z^i = \sum_{j,l,m} Z_{jlm}Y^i_{jlm}, \qquad Z_{jlm} = (-1)^{j-l+m+1}Z^*_{jlm}, \tag{A.3}$$

to find that the Z^i part of the action (3.17) becomes

$$\frac{1}{2}\mathrm{Tr}\Bigg\{\left|\dot Z_{j-1,jm}\right|^2 + \left|\dot Z_{j,jm}\right|^2 + \left|\dot Z_{j+1,jm}\right|^2 - \left(\frac{1}{3}\right)^2(j+1-\eta_s)^2\left|Z_{j-1,jm}\right|^2 -$$

$$- \left(\frac{1}{3}\right)^2(1-\eta_s)^2\left|Z_{jjm}\right|^2 - \left(\frac{1}{3}\right)^2(j+\eta_s)^2\left|Z_{j+1,jm}\right|^2 - \left(\frac{1}{3}\right)^2 j(j+1)\left|Z_{jjm}\right|^2 -$$

$$- (1-\eta_s)\left(\frac{1}{3}\right)^2(j+1)\left|Z_{j-1,jm}\right|^2 - (1-\eta_s)\left(\frac{1}{3}\right)^2\left|Z_{jjm}\right|^2 +$$

$$+ (1-\eta_s)\left(\frac{1}{3}\right)^2 j\left|Z_{j-1,jm}\right|^2\Bigg\}. \tag{A.4}$$

It is then easy to read off the masses of the eigenmodes. For $\eta_s = 1$,

$$\text{for } Z_{j-1,j,m}, \quad \text{the mass is } \frac{1}{3}j,$$

$$\text{for } Z_{jjm}, \qquad \text{the mass is } \frac{1}{3}\sqrt{j(j+1)},$$

$$\text{for } Z_{j+1,jm}, \quad \text{the mass is } \frac{1}{3}(j+1), \tag{A.5}$$

with the degeneracies given in eq. (A.1). (This spectrum agrees with that given in [12], of course.) For $\eta_s = -1$,

$$\text{for } Z_{j-1,jm}, \quad \text{the mass is } \frac{1}{3}\sqrt{j^2 + 6j + 6},$$

$$\text{for } Z_{j,jm}, \qquad \text{the mass is } \frac{1}{3}\sqrt{j^2 + j + 6},$$

$$\text{for } Z_{j+1,jm}, \quad \text{the mass is } \frac{1}{3}\sqrt{j^2 - 4j + 1}, \tag{A.6}$$

with the degeneracies again given in eq. (A.1).

B. Finding the masses of $\Phi^{1,2,3}$ — fuzzy spherical harmonics for anticommutators

The generalized Jacobi identities

$$\{A, \{B, C\}\} = \{\{A, B\}, C\} - [B, [A, C]] = [[A, B], C] + \{B, \{A, C\}\}, \tag{B.1}$$

will be used heavily in this appendix.

– 18 –

B.1 A comment on fuzzy spherical harmonics

Fuzzy spherical harmonics have been reviewed extensively in [22], and we will not repeat those comments here. However, we will change notation relative to that reference, so that now the SU(2) index is $i, j, \cdots = 1, 2, 3$. We trust the reader will not get too confused by the use of j, l, m as both indices and angular momentum quantum numbers.

As a consequence of identity (A.32) of [22] (which expands a product of two spherical harmonics as a sum of spherical harmonics), observe that since (cf.. equation (A.16) of [22])

$$J^{\pm} = \mp\sqrt{\frac{N^2-1}{6}}Y_{1,\pm 1}, \qquad J^3 = \sqrt{\frac{N^2-1}{12}}Y_{10}, \tag{B.2}$$

the following identities hold:

$$J^3 Y_{\ell m}^{(N_1,N_2)} = \frac{\sqrt{[(\ell+1)^2-m^2][(\ell+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(\ell+1)^2]}}{2(\ell+1)\sqrt{(2\ell+1)(2\ell+3)}}Y_{\ell+1,m}+$$

$$+\frac{m[\ell(\ell+1)+\frac{N_1^2-N_2^2}{4}]}{2\ell(\ell+1)}Y_{\ell m}+$$

$$+\frac{\sqrt{(\ell^2-m^2)[\ell^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-\ell^2]}}{2\ell\sqrt{(2\ell+1)(2\ell-1)}}Y_{\ell-1,m}, \tag{B.3a}$$

$$J^+ Y_{\ell m}^{(N_1,N_2)} = -\frac{\sqrt{(\ell+m+1)(\ell+m+2)[(\ell+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(\ell+1)^2]}}{2(\ell+1)\sqrt{(2\ell+1)(2\ell+3)}}Y_{\ell+1,m+1}+$$

$$+\frac{\sqrt{(\ell-m)(\ell+m+1)}[\ell(\ell+1)+\frac{N_1^2-N_2^2}{4}]}{2\ell(\ell+1)}Y_{\ell,m+1}+$$

$$+\frac{\sqrt{(\ell-m-1)(\ell-m)[\ell^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-\ell^2]}}{2\ell\sqrt{(2\ell+1)(2\ell-1)}}Y_{\ell-1,m+1}. \tag{B.3b}$$

This, the known commutators, and hermitian conjugation, is sufficient to determine,

$$\left\{J^3, Y_{\ell m}^{(N_1,N_2)}\right\} = \frac{\sqrt{[(\ell+1)^2-m^2][(\ell+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(\ell+1)^2]}}{(\ell+1)\sqrt{(2\ell+1)(2\ell+3)}}Y_{\ell+1,m}+$$

$$+\frac{m[\frac{N_1^2-N_2^2}{4}]}{\ell(\ell+1)}Y_{\ell m}+$$

$$+\frac{\sqrt{(\ell^2-m^2)[\ell^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-\ell^2]}}{\ell\sqrt{(2\ell+1)(2\ell-1)}}Y_{\ell-1,m}, \tag{B.4a}$$

$$\left\{J^{\pm}, Y_{\ell m}^{(N_1,N_2)}\right\} = \mp\frac{\sqrt{(\ell\pm m+1)(\ell\pm m+2)[(\ell+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(\ell+1)^2]}}{(\ell+1)\sqrt{(2\ell+1)(2\ell+3)}}Y_{\ell+1,m\pm 1}$$

$$+\frac{\sqrt{(\ell\mp m)(\ell\pm m+1)}[\frac{N_1^2-N_2^2}{4}]}{\ell(\ell+1)}Y_{\ell,m\pm 1}\pm$$

$$\pm\frac{\sqrt{(\ell\mp m-1)(\ell\mp m)[\ell^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-\ell^2]}}{\ell\sqrt{(2\ell+1)(2\ell-1)}}Y_{\ell-1,m\pm 1}. \tag{B.4b}$$

We will want these identities shortly.

B.2 Fuzzy spherical vector eigenvectors

The action (3.32) has equation of motion

$$\ddot{\Phi}^i = -\frac{N_1^2 + N_2^2}{18}\Phi^i - \left(\frac{\chi^i}{\mu}\right)^2 \Phi^i + \frac{1}{9}\left[J^j,\left[J^j,\Phi^i\right]\right] + \frac{2}{9}i\epsilon^{ijk}\left[J^j,\Phi^k\right] - \frac{1}{3}i\epsilon^{ijk}\left\{J^j,\Phi^k\right\}. \quad \text{(B.5)}$$

So it is sufficient to consider the eigenvalue equation

$$\left[J^j,\left[J^j,\Phi^i\right]\right] + 2i\epsilon^{ijk}\left[J^j,\Phi^k\right] - 3i\epsilon^{ijk}\left\{J^j,\Phi^k\right\} = \lambda\Phi^i. \quad \text{(B.6)}$$

This is what we do now.

Let us attempt to analyse this by expanding Φ^i in the (ordinary) fuzzy vector spherical harmonics. To do this, we need to know

$$\epsilon^{ijk}\left\{J^j,Y_{j\ell m}^k\right\}. \quad \text{(B.7)}$$

We will work these out, in terms of vector spherical harmonics, in turn.

B.2.1 Preliminary mathematical results

Before starting, we can obtain a useful fact, namely, that the inner products of these vectors is

$$\frac{1}{\sqrt{N_1 N_2}}\operatorname{Tr}\epsilon^{ijk}\left\{J^j,Y_{j\ell m}^k\right\}\epsilon^{ilm}\left\{J^l,Y_{j'\ell'm'}^{m\dagger}\right\} =$$

$$= \left[\frac{N_1^2 + N_2^2}{2} + \lambda_{j\ell}\right]\delta_{jj'}\delta_{\ell\ell'}\delta_{mm'} - \frac{1}{\sqrt{N_1 N_2}}\operatorname{Tr}\left\{J^i,Y_{j\ell m}^i\right\}\left\{J^j,Y_{j'\ell'm'}^{j\dagger}\right\},$$

$$\lambda_{j,\ell} = \begin{cases} -j^2, & j = \ell+1, \\ -j(j+1), & j = \ell, \\ -j^2 - j - 1, & j = \ell-1. \end{cases} \quad \text{(B.8)}$$

This result is obtained using generalized Jacobi identities and the properties (A.34)-(A.38) of [22]. Also,

$$\left[J^l,\left[J^l,\epsilon^{ijk}\left\{J^j,Y_{j\ell m}^k\right\}\right]\right] = 2i\left[J^i,\left\{J^j,Y_{jlm}^j\right\}\right] - 2i\sqrt{j(j+1)}\delta_{j,\ell}\left\{J^i,Y_{jm}\right\} +$$

$$+ \ell(\ell+1)\epsilon^{ijk}\left\{J^l,Y_{j\ell m}^m\right\}, \quad \text{(B.9)}$$

upon using generalized Jacobi identities and (A.34)-(A.38) of [22]. So we see that knowing $\left\{J^i,Y_{j\ell m}^i\right\}$ gives us valuable clues to learning $\epsilon^{ijk}\left\{J^j,Y_{j\ell m}^k\right\}$.

In fact, it is easy to evaluate

$$\left\{J^i,Y_{jjm}^i\right\} = \frac{1}{\sqrt{j(j+1)}}\left[\boldsymbol{J}^2,Y_{jm}\right] = \frac{N_1^2 - N_2^2}{4\sqrt{j(j+1)}}Y_{jm}. \quad \text{(B.10)}$$

A more tedious calculation, using (cf. the signs of eq. (A.4) in [22]) $\boldsymbol{J} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{e}}_{-1}J^+ - \frac{1}{\sqrt{2}}\hat{\boldsymbol{e}}_{+1}J^- + \hat{\boldsymbol{e}}_0 J^3$ and (B.4) is

$$\left\{J^i,Y_{j+1,j,m}^i\right\} = \left[\frac{\sqrt{(j-m)(j-m+1)}}{2\sqrt{(j+1)(2j+1)}}\left\{J^-,Y_{j,m+1}\right\} -\right.$$

j, ℓ, m	$\left\{ J^i, Y^i_{j\ell m} \right\}$
$j+1, j, m$	$\dfrac{\sqrt{[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{\sqrt{(j+1)(2j+3)}} Y_{j+1,m}$
j, j, m	$\dfrac{N_1^2 - N_2^2}{4\sqrt{j(j+1)}} Y_{jm}$
$j-1, j, m$	$-\dfrac{\sqrt{[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{\sqrt{j(2j-1)}} Y_{j-1,m}$

Table 2: Expressions for $\left\{ J^i, Y^i_{j\ell m} \right\}$.

$$
\left. - \frac{\sqrt{(j+m)(j+m+1)}}{2\sqrt{(j+1)(2j+1)}} \left\{ J^+, Y_{j,m+1} \right\} + \frac{\sqrt{(j+1)^2 - m^2}}{\sqrt{(j+1)(2j+1)}} \left\{ J^3, Y_{jm} \right\} \right]
$$
$$
= \frac{\sqrt{[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{\sqrt{(j+1)(2j+3)}} Y_{j+1,m}. \tag{B.11}
$$

Similarly,

$$
\left\{ J^i, Y^i_{j-1,j,m} \right\} = - \frac{\sqrt{[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{\sqrt{j(2j-1)}} Y_{j-1,m}. \tag{B.12}
$$

These are summarized in table 2. The results of table 2, inserted into eq. (B.8), now allow us to tabulate the inner products $\frac{1}{\sqrt{N_1 N_2}} \mathrm{Tr}\, \epsilon^{ijk} \left\{ J^j, Y^k_{j\ell m} \right\} \epsilon^{ilm} \left\{ J^l, Y^{m\dagger}_{j'\ell'm'} \right\}$. These are given in table 3.

The results of table 2, inserted into (B.9), yield,

$$
\left[J^l, \left[J^l, \epsilon^{ijk} \left\{ J^j, Y^k_{j+1,jm} \right\} \right] \right] =
$$
$$
= 2i \sqrt{\frac{j+2}{2j+3}} \sqrt{[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]} Y^i_{j+1,j+1,m}
$$
$$
+ j(j+1)\epsilon^{ijk} \left\{ J^j, Y^k_{j+1,j,m} \right\}, \tag{B.13}
$$

and

$$
\left[J^l, \left[J^l, \epsilon^{ijk} \left\{ J^j, Y^k_{j-1,jm} \right\} \right] \right] =
$$
$$
= -2i \sqrt{\frac{j-1}{2j+1}} \sqrt{[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]} Y^i_{j+1,j+1,m} +
$$
$$
+ j(j+1)\epsilon^{ijk} \left\{ J^j, Y^k_{j-1,j,m} \right\}. \tag{B.14}
$$

B.2.2 The antisymmetric anticommutators of vector spherical harmonics

Now let us start working out the anticommutators (B.7). Note that

$$
\epsilon^{ijk} \left\{ J^j, Y^k_{jjm} \right\} = \frac{2}{\sqrt{j(j+1)}} i \left\{ J^i, Y_{jm} \right\} - \frac{1}{\sqrt{j(j+1)}} \epsilon^{ijk} \left[J^j, \left\{ J^k, Y_{jm} \right\} \right]. \tag{B.15}
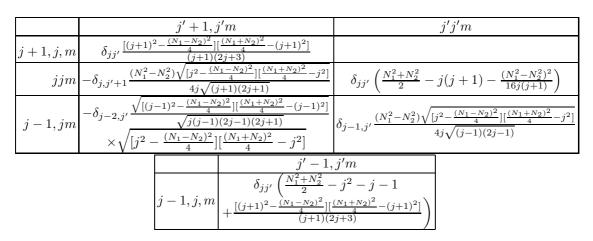$$

	$j'+1, j'm$	$j'j'm$
$j+1, j, m$	$\delta_{jj'}\dfrac{[(j+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(j+1)^2]}{(j+1)(2j+3)}$	
jjm	$-\delta_{j,j'+1}\dfrac{(N_1^2-N_2^2)\sqrt{[j^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-j^2]}}{4j\sqrt{(j+1)(2j+1)}}$	$\delta_{jj'}\left(\dfrac{N_1^2+N_2^2}{2}-j(j+1)-\dfrac{(N_1^2-N_2^2)^2}{16j(j+1)}\right)$
$j-1, jm$	$-\delta_{j-2,j'}\dfrac{\sqrt{[(j-1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(j-1)^2]}}{\sqrt{j(j-1)(2j-1)(2j+1)}}$ $\times\sqrt{[j^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-j^2]}$	$\delta_{j-1,j'}\dfrac{(N_1^2-N_2^2)\sqrt{[j^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-j^2]}}{4j\sqrt{(j-1)(2j-1)}}$

	$j'-1, j'm$
$j-1, j, m$	$\delta_{jj'}\left(\dfrac{N_1^2+N_2^2}{2}-j^2-j-1 + \dfrac{[(j+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(j+1)^2]}{(j+1)(2j+3)}\right)$

Table 3: Expressions for $\frac{1}{\sqrt{N_1 N_2}}\operatorname{Tr}\epsilon^{ijk}\left\{J^j, Y^k_{j\ell m}\right\}\epsilon^{ilm}\left\{J^l, Y^{m\dagger}_{j'\ell'm'}\right\}$ all vanish unless $m = m'$. Note that the table is essentially symmetric; the blank entries can be deduced from the rest of the table.

Given eq. (A.1), it is therefore sufficient to determine $\left\{J^i, Y_{jm}\right\}$. Moreover, it is easy to work out the inner product of the latter with $Y^i_{j'j'm'}$:

$$
\begin{aligned}
\frac{1}{\sqrt{N_1 N_2}}\operatorname{Tr}Y^{i\dagger}_{j'j'm'}\left\{J^i, J_{jm}\right\} &= \frac{1}{\sqrt{N_1 N_2}\,j(j+1)}\operatorname{Tr}\left\{J^i, \left[J^i, Y^\dagger_{j'm'}\right]\right\}Y_{jm} \\
&= \frac{1}{\sqrt{N_1 N_2}\,j(j+1)}\operatorname{Tr}\left[\boldsymbol{J}^2, Y^\dagger_{j'm'}\right]Y_{jm} \\
&= \frac{N_1^2 - N_2^2}{4\sqrt{j(j+1)}}\delta_{jj'}\delta_{mm'}.
\end{aligned}
\tag{B.16}
$$

Comparing the spherical harmonics in (B.4) to those in (A.33) of [22], it is straightforward to see that at most $\left\{J^i, Y_{jm}\right\}$ also has pieces proportional to $Y^i_{j+2,j+1,m}$, $Y^i_{j+1,j,m}$, $Y^i_{j,j-1,m}$, $Y^i_{j,j+1,m}$, $Y^i_{j-1,j,m}$ and $Y^i_{j-2,j-1,m}$. Matching coefficients yields only

$$
\left\{J^i, Y_{jm}\right\} = \frac{\sqrt{[j^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-j^2]}}{\sqrt{j(2j+1)}}Y^i_{j,j-1,m} + \frac{N_1^2-N_2^2}{4\sqrt{j(j+1)}}Y^i_{jjm} - \\
- \frac{\sqrt{[(j+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(j+1)^2]}}{\sqrt{(2j+1)(j+1)}}Y^i_{j,j+1,m}.
\tag{B.17}
$$

Thus,

$$
\epsilon^{ijk}\left\{J^j, Y^k_{jjm}\right\} = i\frac{\sqrt{(j+1)[j^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-j^2]}}{j\sqrt{(2j+1)}}Y^i_{j,j-1,m} + i\frac{N_1^2-N_2^2}{4j(j+1)}Y^i_{jjm} + \\
+ i\frac{\sqrt{j[(j+1)^2-\frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4}-(j+1)^2]}}{(j+1)\sqrt{(2j+1)}}Y^i_{j,j+1,m}.
\tag{B.18}
$$

Now let us evaluate

$$
\epsilon^{ijk}\left\{J^j, Y^k_{j+1,j,m}\right\}.
\tag{B.19}
$$

– 22 –

Eq. (B.18) immediately allows us to evaluate, [transferring the anticommutator to $Y_{j'j'm'}^{i\dagger}$ gives a minus sign from the ϵ, but there is another minus sign upon using $\operatorname{Tr} A^{\dagger}B = (\operatorname{Tr} B^{\dagger}A)^*$]

$$\frac{1}{\sqrt{N_1 N_2}} \operatorname{Tr} Y_{j'j'm'}^{i\dagger} \epsilon^{ijk} \left\{ J^j, Y_{j+1,j,m}^k \right\} =$$
$$= i\delta_{j,j'-1}\delta_{mm'} \frac{\sqrt{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+3)}}, \quad \text{(B.20)}$$

and so, schematically,

$$\epsilon^{ijk} \left\{ J^j, Y_{j+1,j,m}^k \right\} = i\frac{\sqrt{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+3)}} \times$$
$$\times Y_{j+1,j+1,m}^i + Y_{j',j'\pm1,m'}^i. \quad \text{(B.21)}$$

Plugging into (B.13), the term proportional to $Y_{j+1,j+1,m}^i$ cancels, so we see that the remaining, schematic terms, have eigenvalue $j(j+1)$ under the action of $[J^i, [J^i, \cdot]]$. This means that

$$\epsilon^{ijk} \left\{ J^j, Y_{j+1,j,m}^k \right\} = i\frac{\sqrt{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+3)}} \times$$
$$\times Y_{j+1,j+1,m}^i + iA_{jm}Y_{j+1,j,m}^i + iB_{jm}Y_{j-1,j,m}^i, \quad \text{(B.22)}$$

with A and B to be determined. Similarly, (B.14) and (B.18) imply

$$\epsilon^{ijk} \left\{ J^j, Y_{j-1,j,m}^k \right\} = i\frac{\sqrt{(j-1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{(2j-1)}} \times$$
$$\times Y_{j-1,j-1,m}^i + iB_{jm}Y_{j+1,j,m}^i + iC_{jm}Y_{j-1,j,m}^i, \quad \text{(B.23)}$$

where the same symmetry that implied the coefficient of $Y_{j-1,j-1,m}^i$ also implies that the value of B is shared.

From table 3, (note that this is satisfied by (B.18)!) we learn that

$$\frac{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}{(j+1)^2(2j+3)} + |A_{jm}|^2 + |B_{jm}|^2 =$$
$$= \frac{N_1^2 + N_2^2}{2} - j^2 - \frac{[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}{(j+1)(2j+3)}, \quad \text{(B.24)}$$

$$\frac{(j-1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}{j^2(2j-1)} + |B_{jm}|^2 + |C_{jm}|^2 =$$
$$= \frac{N_1^2 + N_2^2}{2} - j^2 - j - 1 - \frac{[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}{j(2j+1)}, \quad \text{(B.25)}$$

$$A_{jm}B_{jm}^* + B_{jm}C_{jm}^* = 0, \tag{B.26}$$

$$\frac{\sqrt{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+3)}}A_{jm}+$$

$$+\frac{[N_1^2 - N_2^2]\sqrt{(j+2)[(j+1)^2 - \frac{(N_1+N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{4(j+1)^2(j+2)\sqrt{2j+3}} =$$

$$= -\frac{[N_1^2 - N_2^2]\sqrt{[(j+1)^2 - \frac{(N_1+N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{4(j+1)\sqrt{(j+2)(2j+3)}}, \tag{B.27}$$

$$\frac{\sqrt{(j-1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{2j-1}}B_{jm} = 0, \tag{B.28}$$

$$\frac{\sqrt{(j-1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{2j-1}}C_{jm}+$$

$$+\frac{[N_1^2 - N_2^2]\sqrt{(j-1)[j^2 - \frac{(N_1+N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{4j^2(j-1)\sqrt{2j-1}} =$$

$$= \frac{[N_1^2 - N_2^2]\sqrt{[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{4j\sqrt{(j-1)(2j-1)}}. \tag{B.29}$$

These are respectively the inner products of $(j+1, j, m)$ with itself; $(j-1, j, m)$ with itself; $(j+1, j, m)$ with $(j-1, j, m)$, $(j+1, j, m)$ with $(j+1, j+1, m)$, $(j+1, j, m)$ with $(j-1, j-1, m)$, and $(j-1, j, m)$ with $(j-1, j-1, m)$. In fact, we only need the last three to find, finally, that

$$\epsilon^{ijk}\left\{J^j, Y_{j+1,j,m}^k\right\} = i\frac{\sqrt{(j+2)[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+3)}}Y_{j+1,j+1,m}^i$$

$$- i\frac{N_1^2 - N_2^2}{4(j+1)}Y_{j+1,j,m}^i,$$

$$\frac{|N_1 - N_2|}{2} \le j \le \frac{N_1 + N_2}{2} - 1, -j-1 \le m \le j+1, \tag{B.30a}$$

$$\epsilon^{ijk}\left\{J^j, Y_{jjm}^k\right\} = i\frac{\sqrt{(j+1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{(2j+1)}}Y_{j,j-1,m}^i + i\frac{N_1^2 - N_2^2}{4j(j+1)}Y_{jjm}^i$$

$$+ i\frac{\sqrt{j[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+1)}}Y_{j,j+1,m}^i,$$

$$\frac{|N_1-N_2|}{2} + \delta_{N_1,N_2} \le j \le \frac{N_1+N_2}{2} - 1,$$
$$-j \le m \le j, \tag{B.30b}$$

$$\epsilon^{ijk}\left\{J^j, Y_{j-1,j,m}^k\right\} = i\frac{\sqrt{(j-1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{(2j-1)}}Y_{j-1,j-1,m}^i +$$

$$+ i\frac{N_1^2 - N_2^2}{4j}Y_{j-1,j,m}^i,$$

$$\frac{|N_1 - N_2|}{2} \leq j \leq \frac{N_1 + N_2}{2} - 1, j \geq 1, -j + 1 \leq m \leq j - 1.$$
(B.30c)

In particular, the action of $\epsilon^{ijk} \{ J^j, \cdot \}$ preserves the jm values of $Y^k_{j\ell m}$! Also, the coefficients of "out-of-range" vector spherical harmonics on the right-hand sides of these equations (e.g. the first term for $j = \frac{1}{2}$ in (B.30b)) vanish.

B.2.3 Eigenvectors of (B.6)

We can now "solve" the eigen problem (B.6). Before presenting the gory details, let us present the solution in a (hopefully) transparent manner. The solutions are labelled[8]

$$\overset{+}{\boldsymbol{Y}}_{njm},$$
(B.31)

with corresponding eigenvalues

$$\lambda_{njm} = (j + 2)(j - 1) + \tilde{\lambda}_{njm}.$$
(B.32)

Generically, $n = -1, 0, 1$, $\frac{|N_1 - N_2|}{2} - 1 \leq j \leq \frac{N_1 + N_2}{2}$ and $-j \leq m \leq j$. However, n does not run over all three values for all j and j does not reach the lower limits for all N_1, N_2. So more precisely, the allowed values of j are

$$
\begin{aligned}
j = 1 &= \frac{N_1 + N_2}{2}, & N_1 = N_2 = 1, \\
\frac{|N_1 - N_2|}{2} = 0 \leq j &\leq N = \frac{N_1 + N_2}{2}, & N_1 = N_2 = N > 1, \\
\frac{|N_1 - N_2|}{2} = \frac{1}{2} \leq j &\leq N \pm \frac{1}{2} = \frac{N_1 + N_2}{2}, & N \equiv N_1 = N_2 \pm 1, \\
\frac{|N_1 - N_2|}{2} - 1 \leq j &\leq \frac{N_1 + N_2}{2}, & N_1 \leq N_2 - 2 \text{ or } N_1 \geq N_2 + 2.
\end{aligned}
$$
(B.33)

That is, the lower limit of j must (not surprisingly) be nonnegative and (more surprisingly, but this follows from the nonexistence of the vector spherical harmonic $\boldsymbol{Y}_{010}$ (in (A.33c) of [22], $\frac{N_1 + N_2}{2} - 2 = -1$ is invalid) for $N_1 = N_2 = 1$) if $N_1 = N_2 = 1$ then $j \neq 0$ as well.

---

[8]Alternatively, we could have chosen to use $\overset{+}{\boldsymbol{Y}}_{j\ell m}$ as for the (ordinary) vector spherical harmonics — and the spinors, but this seemed slightly unnatural.

(a)

$N_1 = N_2 = 1$
$j = 1$

$j = 1$	$n = -1$
	$\tilde{\lambda}_{-1,1,m} = 0$

(b)

$N_1 = 1, N_2 = 2$
$\frac{1}{2} \le j \le \frac{3}{2}$

$j = \frac{1}{2}$	$n = 0$
	$\tilde{\lambda}_{0,\frac{1}{2},m} = -3$
$j = \frac{3}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{3}{2},m} = \frac{3}{2}$

(c)

$N_1 = 2, N_2 = 1$
$\frac{1}{2} \le j \le \frac{3}{2}$

$j = \frac{1}{2}$	$n = 0$
	$\tilde{\lambda}_{0,\frac{1}{2},m} = 3$
$j = \frac{3}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{3}{2},m} = -\frac{3}{2}$

(d)

$N_1 = 1, N_2 > 2$
$\frac{N_2-1}{2} - 1 \le j \le \frac{N_2+1}{2}$

$j = \frac{N_2-3}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{N_2-3}{2},m} = -\frac{3}{2}(N_2 + 1)$
$j = \frac{N_2-1}{2}$	$n = 0$
	$\tilde{\lambda}_{0,\frac{N_2-1}{2},m} = -3$
$j = \frac{N_2+1}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{N_2+1}{2},m} = \frac{3}{2}(N_2 - 1)$

(e)

$N_1 > 2, N_1 = 1$
$\frac{N_1-1}{2} - 1 \le j \le \frac{N_1+1}{2}$

$j = \frac{N_1-3}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{N_1-3}{2},m} = \frac{3}{2}(N_1 + 1)$
$j = \frac{N_1-1}{2}$	$n = 0$
	$\tilde{\lambda}_{0,\frac{N_1-1}{2},m} = 3$
$j = \frac{N_1+1}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,\frac{N_1+1}{2},m} = -\frac{3}{2}(N_1 - 1)$

(f)

$N \equiv N_1 = N_2 - 1 > 1$
$\frac{1}{2} \le j \le N + \frac{1}{2}$

$j = \frac{1}{2}$	$n = 0, 1$
	$\tilde{\lambda}_{0,\frac{1}{2},m} = -\frac{3}{4}(2N+1) + \frac{3}{4}\sqrt{4N^2 + 4N - 7}$
	$\tilde{\lambda}_{1,\frac{1}{2},m} = -\frac{3}{4}(2N+1) - \frac{3}{4}\sqrt{4N^2 + 4N - 7}$
$\frac{3}{2} \le j \le N - \frac{3}{2}$	$n = -1, 0, 1$
	$\tilde{\lambda}_{njm} =$ eq. (B.40)
$j = N - \frac{1}{2}$	$n = 0, 1$
	$\tilde{\lambda}_{0,N-\frac{1}{2},m} = \frac{3}{4} + \frac{3}{4}\sqrt{16N + 9}$
	$\tilde{\lambda}_{1,N-\frac{1}{2},m} = \frac{3}{4} - \frac{3}{4}\sqrt{16N + 9}$
$j = N + \frac{1}{2}$	$n = -1$
	$\tilde{\lambda}_{-1,N+\frac{1}{2},m} = \frac{3}{2}$

Table 4: The allowed values of j and n, and the corresponding eigenvalues for the various cases. It should be noted that although the values of the eigenvalues are always obtainable from eq. (B.40), the corresponding values of n are not the same between the conventions here and there, except when n is allowed to take on all three values. This table is continued...

$N_1 = N_2 + 1 \equiv N + 1 > 2$ $\frac{1}{2} \le j \le N + \frac{1}{2}$	
$j = \frac{1}{2}$	$n = 0, 1$
	$\lambda_{0,\frac{1}{2},m} = \frac{3}{4}(2N+1) + \frac{3}{4}\sqrt{4N^2 + 4N - 7}$
	$\tilde\lambda_{1,\frac{1}{2},m} = \frac{3}{4}(2N+1) - \frac{3}{4}\sqrt{4N^2 + 4N - 7}$
$\frac{3}{2} \le j \le N - \frac{3}{2}$	$n = -1, 0, 1$
	$\lambda_{njm} = $ eq. (B.40)
$j = N - \frac{1}{2}$	$n = 0, 1$
	$\tilde\lambda_{0,N-\frac{1}{2},m} = -\frac{3}{4} + \frac{3}{4}\sqrt{16N + 9}$
	$\tilde\lambda_{1,N-\frac{1}{2},m} = -\frac{3}{4} - \frac{3}{4}\sqrt{16N + 9}$
$j = N + \frac{1}{2}$	$n = -1$
	$\lambda_{-1,N+\frac{1}{2},m} = -\frac{3}{2}$

(g)

$N_1 = N_2 = N > 1$ $0 \le j \le N$	
$j = 0$	$n = -1$
	$\lambda_{-1,0,0} = 0$
$1 \le j \le N - 2$	$n = -1, 0, 1$
	$\lambda_{-1,j,m} = 0$
	$\tilde\lambda_{0,j,m} = 3\sqrt{N^2 - j(j+1)}$
	$\tilde\lambda_{1,j,m} = -3\sqrt{N^2 - j(j+1)}$
$j = N - 1$	$n = 0, 1$
	$\tilde\lambda_{0,N-1,m} = 3\sqrt{N}$
	$\tilde\lambda_{1,N-1,m} = -3\sqrt{N}$
$j = N$	$n = -1$
	$\tilde\lambda_{-1,N,m} = 0$

(h)

$|N_1 - N_2| > 2; N_1, N_2 > 1$ $\frac{|N_1-N_2|}{2} - 1 \le j \le \frac{N_1+N_2}{2}$	
$j = \frac{|N_1-N_2|}{2} - 1$	$n = -1$
	$\lambda_{-1,\frac{|N_1-N_2|}{2}-1,m} = \frac{3}{2}\,\text{sgn}(N_1-N_2)(N_1+N_2)$
$j = \frac{|N_1-N_2|}{2}$	$n = 0, 1$
	$\lambda_{0,\frac{|N_1-N_2|}{2},m} = \frac{3}{4}\,\text{sgn}(N_1-N_2)(N_1+N_2) + \frac{3}{4}\sqrt{(N_1+N_2)^2 - 8\,|N_1 - N_2|}$
	$\tilde\lambda_{1,\frac{|N_1-N_2|}{2},m} = \frac{3}{4}\,\text{sgn}(N_1-N_2)(N_1+N_2) - \frac{3}{4}\sqrt{(N_1+N_2)^2 - 8\,|N_1 - N_2|}$
$\frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2$	$n = -1, 0, 1$
	$\lambda_{n,j,m} = $ eq. (B.40)
$j = \frac{N_1+N_2}{2} - 1$	$n = 0, 1$
	$\lambda_{0,\frac{N_1+N_2}{2}-1,m} = -\frac{3}{4}(N_1-N_2) + \frac{3}{4}\sqrt{(N_1-N_2)^2 + 8(N_1+N_2)}$
	$\tilde\lambda_{0,\frac{N_1+N_2}{2}-1,m} = -\frac{3}{4}(N_1-N_2) - \frac{3}{4}\sqrt{(N_1-N_2)^2 + 8(N_1+N_2)}$
$j = \frac{N_1+N_2}{2}$	$n = -1$
	$\tilde\lambda_{-1,\frac{N_1+N_2}{2},m} = -\frac{3}{2}(N_1-N_2)$

(i)

Table 4: ... The rest of the table.

The range of n depends on j; as the range of j depends on N_1 and N_2, it might be most transparent to separate out those cases at the risk of redundancy. See also table 4. Explicitly,

$$\frac{|N_1 - N_2|}{2} \le j \le \frac{N_1 + N_2}{2}, m = -j \ldots j,$$

$$n = \begin{cases} -1, 0, 1, & \frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2, \\ -1, & j = \frac{N_1+N_2}{2}, \\ 0, 1 & j = \frac{N_1+N_2}{2} - 1, \\ 0, 1 & j = \frac{|N_1-N_2|}{2} > 0 \text{ and } N_1, N_2 > 1, \\ 0 & 0 < j = \frac{|N_1-N_2|}{2} \text{ and } (N_1 = 1 \text{ or } N_2 = 1), \\ -1, & j = 0, (\text{i.e. } j = \frac{|N_1-N_2|}{2} \text{ and } N_1 = N_2), \\ -1, & j = \frac{|N_1-N_2|}{2} - 1 \ge 0. \end{cases} \tag{B.34}$$

That is, generically, $n = 0, \pm 1$, but not all values of j allow for all three eigenvectors. This is particularly complicated as one must watch for possible overlap of the various naïvely different cases; this is why there is a difference, for example, between having $N_1 = 1$ or $N_2 = 1$ and having $N_1, N_2 > 1$.

The $\overset{+}{\boldsymbol{Y}}_{njm}$ are constructed to be (normalized) eigensolutions to (B.6), with eigenvalue λ_{njm}, given by

$$\lambda_{njm} = (j + 2)(j - 1) + \tilde{\lambda}_{njm}, \tag{B.35}$$

where,

$$\tilde{\lambda}_{njm} = \begin{cases} \sqrt{6}\sqrt{N_1^2 + N_2^2 - 2j(j+1)} \times & n = 0, \pm 1; \\ \times \cos\left[\frac{\pi(2n+1)}{3} - \frac{1}{3}\cos^{-1}\sqrt{\frac{27}{8}\frac{N_1^2 - N_2^2}{[N_1^2 + N_2^2 - 2j(j+1)]^{3/2}}}\right], & \frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2, \\ -\frac{3}{2}(N_1 - N_2), & n = -1; j = \frac{N_1+N_2}{2}. \end{cases} \tag{B.36}$$

Given that the commutators preserve the vector spherical harmonics, and $\epsilon^{abc}\{J^b, \cdot\}$ preserves their first and last index, we can take

$$\Phi^a = \begin{cases} \alpha_{j,j-1}Y^a_{j,j-1,m} + \alpha_{j,j}Y^a_{jjm} + \alpha_{j,j+1}Y^a_{j,j+1,m}, & \frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2, \\ \alpha_{j,j}Y^a_{jjm} + \alpha_{j,j+1}Y^a_{j,j+1,m}, & N_1 \ne N_2, \min(N_1, N_2) \ne 1, j = \frac{|N_1-N_2|}{2}, \\ Y^a_{\frac{|N_1-N_2|}{2}\frac{|N_1-N_2|}{2}m}, & \min(N_1, N_2) = 1, N_1 \ne N_2, j = \frac{|N_1-N_2|}{2} \\ Y^a_{j,j+1,m}, & j = \frac{|N_1-N_2|}{2} - 1 \ge 0, \\ Y^a_{010}, & N_1 = N_2, j = 0, \\ \alpha_{j,j-1}Y^a_{j,j-1,m} + \alpha_{j,j}Y^a_{jjm}, & j = \frac{N_1+N_2}{2} - 1, \min(N_1, N_2) > 1, \\ Y^a_{j,j-1,m}, & j = \frac{N_1+N_2}{2}, \end{cases}$$
$$- j \le m \le j. \tag{B.37}$$

(This indeed gives $3N_1N_2$ possibilities. The constraint on the second line, and its complement on the third line, comes from the upper bound on j in eq. (B.30c). There is

a similar constraint on the second-last line, with no additional complement; indeed, if $\min(N_1, N_2) = 1$ then $\frac{N_1+N_2}{2} - 1 = \frac{|N_1-N_2|}{2}$, and so not only would the second-last line be redundant with the third line, but eq. (B.30a) shows that there is no $Y^i_{j,j-1,m}$ for this value of j.) Then (A.35), (A.37) of [22] and (B.30) transform the eigenvalue problem (B.6) into the eigenvalue problems,

$$
\begin{pmatrix}
(j+2)(j-1) - \frac{3}{4}\frac{N_1^2-N_2^2}{j} & 3\frac{\sqrt{(j+1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{(2j+1)}} & 0 \\
3\frac{\sqrt{(j+1)[j^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{j\sqrt{(2j+1)}} & (j+2)(j-1) + \frac{3}{4}\frac{N_1^2-N_2^2}{j(j+1)} & 3\frac{\sqrt{j[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+1)}} \\
0 & 3\frac{\sqrt{j[(j+1)^2 - \frac{(N_1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{(j+1)\sqrt{(2j+1)}} & (j+2)(j-1) + \frac{3}{4}\frac{N_1^2-N_2^2}{j+1}
\end{pmatrix} \times
$$
$$
\times
\begin{pmatrix} \alpha_{j,j-1} \\ \alpha_{j,j} \\ \alpha_{j,j+1} \end{pmatrix}
= \lambda_j \begin{pmatrix} \alpha_{j,j-1} \\ \alpha_{j,j} \\ \alpha_{j,j+1} \end{pmatrix},
\frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2.
\tag{B.38a}
$$

$$
\begin{pmatrix}
\frac{(N_1-N_2)^2}{4} + \frac{|N_1-N_2|}{2} - 2 + 3\,\mathrm{sgn}(N_1 - N_2)\frac{N_1+N_2}{|N_1-N_2|+2} & 3\frac{\sqrt{2|N_1-N_2|[N_1N_2 - |N_1-N_2|-1]}}{|N_1-N_2|+2} \\
3\frac{\sqrt{2|N_1-N_2|[N_1N_2 - |N_1-N_2|-1]}}{|N_1-N_2|+2} & \frac{(N_1-N_2)^2}{4} + \frac{|N_1-N_2|}{2} - 2 + \frac{3}{2}\frac{N_1^2-N_2^2}{|N_1-N_2|+2}
\end{pmatrix}
$$
$$
\times
\begin{pmatrix} \alpha_{\frac{|N_1-N_2|}{2}, \frac{|N_1-N_2|}{2}} \\ \alpha_{\frac{|N_1-N_2|}{2}, \frac{|N_1-N_2|}{2}+1} \end{pmatrix}
= \lambda_{\frac{|N_1-N_2|}{2}} \begin{pmatrix} \alpha_{\frac{|N_1-N_2|}{2}, \frac{|N_1-N_2|}{2}} \\ \alpha_{\frac{|N_1-N_2|}{2}, \frac{|N_1-N_2|}{2}+1} \end{pmatrix},
j = \frac{|N_1 - N_2|}{2} \ne 0, \min(N_1, N_2) \ne 1,
\tag{B.38b}
$$

$$
\left(\frac{(N_1-N_2)^2}{4} + \frac{|N_1-N_2|}{2} - 2 + 3\,\mathrm{sgn}(N_1 - N_2)\frac{N_1+N_2}{|N_1-N_2|+2}\right) \left(\alpha_{\frac{|N_1-N_2|}{2}\,\frac{|N_1-N_2|}{2}}\right)
$$
$$
= \lambda_{\frac{|N_1-N_2|}{2}} \left(\alpha_{\frac{|N_1-N_2|}{2}\,\frac{|N_1-N_2|}{2}}\right), j = \frac{|N_1 - N_2|}{2}, \min(N_1, N_2) = 1, N_1 \ne N_2,
\tag{B.38c}
$$

$$
\left(\frac{(N_1-N_2)^2}{4} - \frac{|N_1-N_2|}{2} - 2 + \frac{3}{2}(N_1 + N_2)\,\mathrm{sgn}(N_1 - N_2)\right) \left(\alpha_{\frac{|N_1-N_2|}{2}-1,\frac{|N_1-N_2|}{2}}\right)
$$
$$
= \lambda_{\frac{|N_1-N_2|}{2}-1} \left(\alpha_{\frac{|N_1-N_2|}{2}-1,\frac{|N_1-N_2|}{2}}\right), \qquad j = \frac{|N_1 - N_2|}{2} - 1 \ge 0,
\tag{B.38d}
$$

$$
(-2)\left(\alpha_{0,1,0}\right) = \lambda_0\left(\alpha_{0,1,0}\right), \qquad N_1 = N_2, j = 0,
\tag{B.38e}
$$

$$
\begin{pmatrix}
\frac{(N_1+N_2)^2}{4} - \frac{N_1+N_2}{2} - 2 - \frac{3}{2}\frac{N_1^2-N_2^2}{N_1+N_2-2} & 3\frac{\sqrt{2(N_1-1)(N_2-1)(N_1+N_2)}}{N_1+N_2-2} \\
3\frac{\sqrt{2(N_1-1)(N_2-1)(N_1+N_2)}}{N_1+N_2-2} & \frac{(N_1+N_2)^2}{4} - \frac{N_1+N_2}{2} - 2 + 3\frac{N_1-N_2}{N_1+N_2-2}
\end{pmatrix}
\begin{pmatrix} \alpha_{\frac{N_1+N_2}{2}-1,\frac{N_1+N_2}{2}-2} \\ \alpha_{\frac{N_1+N_2}{2}-1,\frac{N_1+N_2}{2}-1} \end{pmatrix}
$$
$$
= \lambda_{\frac{N_1+N_2}{2}-1} \begin{pmatrix} \alpha_{\frac{N_1+N_2}{2}-1,\frac{N_1+N_2}{2}-2} \\ \alpha_{\frac{N_1+N_2}{2}-1,\frac{N_1+N_2}{2}-1} \end{pmatrix}, \qquad j = \frac{N_1 + N_2}{2} - 1 \ge 0,
\tag{B.38f}
$$

$$
\left(\frac{(N_1+N_2)^2}{4} + \frac{N_1+N_2}{2} - 2 - \frac{3}{2}(N_1 - N_2)\right) \left(\alpha_{\frac{N_1+N_2}{2},\frac{N_1+N_2}{2}-1}\right) = \lambda_{\frac{N_1+N_2}{2}} \left(\alpha_{\frac{N_1+N_2}{2},\frac{N_1+N_2}{2}-1}\right),
$$
$$
j = \frac{N_1 + N_2}{2}.
\tag{B.38g}
$$

For the generic ($\frac{|N_1-N_2|}{2} + 1 \le j \le \frac{N_1+N_2}{2} - 2$) problem, the eigenvalues λ are given by

$$
\lambda = (j+2)(j-1) + \tilde{\lambda},
\tag{B.39}
$$

where

$$\tilde{\lambda} = \sqrt{6}\sqrt{N_1^2 + N_2^2 - 2j(j+1)}\cos\left[\frac{\pi(2n+1)}{3} - \frac{1}{3}\cos^{-1}\sqrt{\frac{27}{8}}\frac{N_1^2 - N_2^2}{[N_1^2 + N_2^2 - 2j(j+1)]^{3/2}}\right],$$

$$n = 0, \pm 1, \tag{B.40}$$

are the roots of

$$\tilde{\lambda}^3 - 9\left[\frac{N_1^2 + N_2^2}{2} - j(j+1)\right]\tilde{\lambda} + \frac{27}{4}(N_1^2 - N_2^2) = 0. \tag{B.41}$$

The eigenvectors — for $N_1 \neq N_2$ — are then given by

$$4\frac{[N_1^2 - N_2^2 - \frac{4}{3}\tilde{\lambda}(j+1)]\sqrt{[j^2 - \frac{(N-1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - j^2]}}{\sqrt{j(2j+1)\kappa}}\boldsymbol{Y}_{j,j-1,m}+$$

$$+\frac{[N_1^2 - N_2^2 - \frac{4}{3}\tilde{\lambda}(j+1)][N_1^2 - N_2^2 + \frac{4}{3}\tilde{\lambda}j]}{\sqrt{j(j+1)\kappa}}\boldsymbol{Y}_{jjm}-$$

$$-4\frac{[N_1^2 - N_2^2 + \frac{4}{3}\tilde{\lambda}j]\sqrt{[(j+1)^2 - \frac{(N-1-N_2)^2}{4}][\frac{(N_1+N_2)^2}{4} - (j+1)^2]}}{\sqrt{(j+1)(2j+1)\kappa}}\boldsymbol{Y}_{j,j+1,m}, \tag{B.42a}$$

$$\kappa \equiv \left[\frac{256}{9}(N_1^2 + N_2^2)j(j+1) - \frac{512}{9}j^2(j+1)^2 - \frac{16}{3}(N_1^2 - N_2^2)^2\right]\tilde{\lambda}^2-$$

$$-\left[\frac{320}{3}(N_1^2 - N_2^2)j(j+1) - \frac{64}{3}(N_1^4 - N_2^4)\right]\tilde{\lambda}$$

$$-16\left[N_1^2 - N_2^2\right]^2\left[j^2 + j + 3 - \frac{N_1^2 + N_2^2}{2}\right]. \tag{B.42b}$$

As checks, note that the $\tilde{\lambda}$'s sum to zero, which agrees with the trace of the matrix of the eigenvalue problem (after subtracting $(j+2)(j-1)\mathbb{1}$). Also, the product of the $\tilde{\lambda}$'s can be evaluated using

$$\prod_{n=-1}^{1}\cos\left(\frac{2\pi n}{3} + \frac{x}{3}\right) = \prod_{n=-1}^{1}\left(\cos\frac{2\pi n}{3}\cos\frac{x}{3} - \sin\frac{2\pi n}{3}\sin\frac{x}{3}\right) = \cos^3\frac{x}{3} - \frac{3}{4}\cos\frac{x}{3} = \frac{1}{4}\cos x. \tag{B.43}$$

Thus, the product of the $\tilde{\lambda}$'s is

$$\prod_{n=-1}^{1}\tilde{\lambda} = 6^{3/2}(N_1^2 + N_2^2 - 2j(j+1)^{3/2}\cos\left[\pi - \cos^{-1}\sqrt{\frac{27}{8}}\frac{N_1^2 - N_2^2}{[N_1^2 + N_2^2 - 2j(j+1)]^{3/2}}\right]$$

$$= -\frac{27}{4}(N_1^2 - N_2^2). \tag{B.44}$$

which agrees with the determinant of the matrix.

When $j = \frac{N_1+N_2}{2} - 1$, the eigenvectors and eigenvalues $\tilde{\lambda}$ of the relevant matrix

$$\begin{pmatrix} -\frac{3}{2}\frac{N_1^2 - N_2^2}{N_1+N_2-2} & 3\frac{\sqrt{2(N_1-1)(N_2-1)(N_1+N_2)}}{N_1+N_2-2} \\ 3\frac{\sqrt{2(N_1-1)(N_2-1)(N_1+N_2)}}{N_1+N_2-2} & +3\frac{N_1-N_2}{N_1+N_2-2} \end{pmatrix}, \tag{B.45}$$

are

$$\sqrt{\frac{1}{2} \pm \frac{\upsilon}{2\nu}} \boldsymbol{Y}_{\frac{N_1+N_2}{2}-1, \frac{N_1+N_2}{2}-2, m} \pm \sqrt{\frac{1}{2} \mp \frac{\upsilon}{2\nu}} \boldsymbol{Y}_{\frac{N_1+N_2}{2}-1, \frac{N_1+N_2}{2}-1, m},$$

$$\tilde{\lambda} = -\frac{3}{2}(N_1 - N_2) \pm \nu, \tag{B.46a}$$

$$\upsilon \equiv -\frac{3}{4}\frac{(N_1 - N_2)(N_1 + N_2 + 2)}{N_1 + N_2 - 2},$$

$$\nu \equiv \frac{3}{4}\sqrt{(N_1^2 - N_2^2)^2 + 8(N_1 + N_2)}. \tag{B.46b}$$

For $N_1 = N_2 = N$, there is an enormous simplification of the generic problem. We find the normalized eigenvectors

$$-\sqrt{\frac{j[N^2 - (j+1)^2]}{(2j+1)[N^2 - j(j+1)]}} \boldsymbol{Y}_{j,j-1,m} + \sqrt{\frac{(j+1)[N^2 - j^2]}{(2j+1)[N^2 - j(j+1)]}} \boldsymbol{Y}_{j,j+1,m},$$

$$\tilde{\lambda} = 0, \tag{B.47a}$$

$$\sqrt{\frac{(j+1)[N^2 - j^2]}{2(2j+1)[N^2 - j(j+1)]}} \boldsymbol{Y}_{j,j-1,m} \pm \frac{1}{\sqrt{2}} \boldsymbol{Y}_{jjm} + \sqrt{\frac{j[N^2 - (j+1)^2]}{2(2j+1)[N^2 - j(j+1)]}} \boldsymbol{Y}_{j,j+1,m},$$

$$\tilde{\lambda} = \pm 3\sqrt{N^2 - j(j+1)}, \tag{B.47b}$$

for $1 \le j \le \frac{N_1+N_2}{2} - 2$, and

$$\boldsymbol{Y}_{010}, \qquad \tilde{\lambda} = 0, \tag{B.47c}$$

$$\frac{1}{\sqrt{2}} \boldsymbol{Y}_{N-1, N-2, m} \pm \frac{1}{\sqrt{2}} \boldsymbol{Y}_{N-1, N-1, m}, \qquad \tilde{\lambda} = \pm 3\sqrt{N}, N \ne 1, \tag{B.47d}$$

$$\boldsymbol{Y}_{N, N-1, m}, \qquad \tilde{\lambda} = 0. \tag{B.47e}$$

C. Finding the masses of χ

Below we give the details of finding the masses for the fermionic off-diagonal fluctuations χ. This part of the action (3.9) is (after integration by parts)

$$(S_{\text{fermion}})_{\text{o.d.}} = 2\,\mathrm{Tr}\left(i\chi^\dagger \dot{\chi} + \frac{1}{3}\chi^\dagger \gamma^i (\chi J_{(2)}^i + J_{(1)}^i \chi) - \frac{x^a}{\mu}\chi^\dagger \gamma^a \chi - i\frac{1}{4}\chi^\dagger \gamma^{123}\chi \right), \tag{C.1}$$

which gives the Dirac equation

$$\left(i\partial_t - \frac{x^b}{\mu}\gamma^b - i\frac{1}{4}\gamma^{123} + \frac{1}{3}\gamma^j\left\{ J^j, \cdot \right\} \right)\chi = 0. \tag{C.2}$$

Squaring the Dirac equation, i.e. acting on the left with the conjugate Dirac operator $\left(-i\partial_t - \frac{x^a}{\mu}\gamma^a - i\frac{1}{4}\gamma^{123} + \frac{1}{3}\gamma^i\left\{ J^i, \cdot \right\} \right)$ gives the "Klein-Gordon" equation

$$\partial_t^2 \chi + i\frac{\upsilon^b}{\mu}\gamma^b \chi + \frac{x^b x^b}{\mu^2}\chi + \left(\frac{1}{4}\right)^2 \chi - i\frac{1}{6}\gamma^{123}\gamma^i\left\{ J^i, \chi \right\} + \left(\frac{1}{3}\right)^2 \gamma^i\gamma^j\left\{ J^i, \left\{ J^j, \chi \right\} \right\} = 0. \tag{C.3}$$

Upon setting $v^b = v\delta^{b9}$ the term $i\frac{v^b}{\mu}\gamma^b\chi$ becomes $i\frac{v}{\mu}\gamma^9\chi$. Since γ^9 commutes with the other gamma matrices, i.e., $\gamma^{123}\gamma^i$ and $\gamma^i\gamma^j$, in the e.o.m., we can first diagonalize w.r.t. γ^9. Upon the projection

$$\chi_\pm \equiv \frac{1\pm\gamma^9}{2}\chi, \tag{C.4}$$

the e.o.m. separates into $\pm$ parts

$$\partial_t^2\chi_\pm \pm i\frac{v}{\mu}\chi_\pm + \frac{x^b x^b}{\mu^2}\chi_\pm + \left(\frac{1}{4}\right)^2\chi_\pm - i\frac{1}{6}\gamma^{123}\gamma^i\left\{J^i,\chi_\pm\right\} + \left(\frac{1}{3}\right)^2\gamma^i\gamma^j\left\{J^i,\left\{J^j,\chi_\pm\right\}\right\} = 0. \tag{C.5}$$

We see that the difference between the $+$ and $-$ components of the e.o.m. is just $\chi_+ \to \chi_-$ and $v \to -v$. Hence in the following let's first concentrate on the χ_+ equation and suppress the $+$ subscript. Readily seen, solutions to the eigenproblem

$$i\gamma^{123}\gamma^i\left\{J^i,\chi\right\} = \lambda\chi, \tag{C.6}$$

(which implies that $\gamma^i\gamma^j\left\{J^i,\left\{J^j,\chi\right\}\right\} = \lambda^2\chi$) diagonalize the e.o.m., giving the mass squared for χ_+ (after Wick rotation $v \to -iv$)

$$m_{\chi_+}^2 = \frac{v}{\mu} + \frac{x^a x^a}{\mu^2} + \left(\frac{\lambda}{3} - \frac{1}{4}\right)^2, \tag{C.7}$$

and similarly the mass-squared for χ_-

$$m_{\chi_+}^2 = -\frac{v}{\mu} + \frac{x^a x^a}{\mu^2} + \left(\frac{\lambda}{3} - \frac{1}{4}\right)^2. \tag{C.8}$$

We first look at the eigenproblem (C.6) for χ_+. Adopting the gamma matrix representation of [26] where

$$\gamma^9 = \begin{pmatrix} I_{8\times8} & 0 \\ 0 & -I_{8\times8} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \tilde{\gamma}^i \\ (\tilde{\gamma}^i)^T & 0 \end{pmatrix}, \tag{C.9}$$

(with $\tilde{\gamma}^i$'s being 8×8 matrices; more specifically, $\tilde{\gamma}^1 = i\tau_2 \otimes i\tau_2 \otimes i\tau_2$, $\tilde{\gamma}^2 = 1\otimes\tau_1\otimes i\tau_2$, and $\tilde{\gamma}^3 = 1\otimes\tau_3\otimes i\tau_2$, with τ_1,τ_2,τ_3 being the standard Pauli matrices), we see that χ_+ is of the form

$$\chi_+ = \begin{pmatrix} \theta_+ \\ 0 \end{pmatrix}, \tag{C.10}$$

where θ_+ is a 8-component spinor. Then eq. (C.6) becomes

$$1\otimes i\tau_2\otimes 1\left\{J^1,\theta_+\right\} - i\tau_2\otimes\tau_1\otimes 1\left\{J^2,\theta_+\right\} - i\tau_2\otimes\tau_3\otimes 1\left\{J^3,\theta_+\right\} = i\lambda\theta_+. \tag{C.11}$$

Similarly χ_- is of the form

$$\chi_- = \begin{pmatrix} 0 \\ \theta_- \end{pmatrix}, \tag{C.12}$$

and the eigen-equation for χ_- in terms of θ_- is the same as eq. (C.11) with $\theta_+ \to \theta_-$. Hence we see that χ_+ and χ_- have the same eigenvalue λ. To solve eq. (C.11) we do the projection

$$\theta_+ = \theta_1 + \theta_2, \text{with } \theta_1 \equiv \frac{1+\tau_2}{2}\otimes 1\otimes 1\ \theta_+,\ \theta_2 \equiv \frac{1-\tau_2}{2}\otimes 1\otimes 1\ \theta_+, \tag{C.13}$$

which diagonalizes the first gamma matrix in the direct-product of three of them in eq. (C.11) (the third gamma matrix is already diagonalized automatically). Hence we can write θ_1 as a two-component spinor acted upon by the second gamma matrix and the eigen-equation becomes

$$\left\{\tau_2 J^1 - \tau_1 J^2 - \tau_3 J^3, \theta_1\right\} = \lambda_1 \theta_1, \tag{C.14}$$

(where we have added the subscript 1 to λ) and note that for each value of λ_1 there is a degeneracy of two θ_1's (coming from the third gamma matrix, recalling that θ_1 has four real degrees of freedom). Similarly, the equation for θ_2 reads

$$\left\{\tau_2 J^1 + \tau_1 J^2 + \tau_3 J^3, \theta_2\right\} = \lambda_2 \theta_2, \tag{C.15}$$

(where we have added the subscript 2 to λ) with a two-fold degeneracy for each value of λ_2. Below let us first solve the equation for θ_1.

Denote the matrix $(\tau_2 J^1 - \tau_1 J^2 - \tau_3 J^3)$ as Δ_1 and act it on θ_1 twice, we get

$$\{\Delta_1, \{\Delta_1, \theta_1\}\} = \lambda_1^2 \theta_1, \tag{C.16}$$

whose l.h.s. after some algebra can be written as

$$2(\Lambda_{(1)} + \Lambda_{(2)})\theta_1 - \left[J^i, \left[J^i, \theta_1\right]\right] + \tau_2 \left[J^1, \theta_1\right] - \tau_1 \left[J^2, \theta_1\right] - \tau_3 \left[J^3, \theta_1\right]. \tag{C.17}$$

Then we write

$$\theta_1 = \begin{pmatrix} \beta \\ \eta \end{pmatrix}, \tag{C.18}$$

expand β, η

$$\beta = \sum_{l=\frac{|N_1-N_2|}{2}}^{\frac{N_1+N_2}{2}-1} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}, \quad \eta = \sum_{l=\frac{|N_1-N_2|}{2}}^{\frac{N_1+N_2}{2}-1} \sum_{m=-l}^{l} \eta_{lm} Y_{lm}, \tag{C.19}$$

and plug them into eq. (C.17). Solving the resulting equation we find the eigenvectors for the operator $\{\Delta_1, \{\Delta_1, \cdot\}\}$, which we summarize below:

$l = \frac{|N_1-N_2|}{2}, \ldots, \frac{N_1+N_2}{2} - 1$ and for any given l there are two cases:

Case A: $\lambda_1^2 = \frac{N_1^2+N_2^2}{2} - (l+1)^2$, which has a degeneracy of $2l+2$, with $2l$ states given by

$$\begin{pmatrix} i\sqrt{\frac{l+m+1}{l-m}} Y_{lm} \\ Y_{l,m+1} \end{pmatrix} \text{ for } m = -l, \ldots, l-1, \tag{C.20}$$

and the other 2 states being

$$\begin{pmatrix} 0 \\ Y_{l,-l} \end{pmatrix} \text{ and } \begin{pmatrix} Y_{ll} \\ 0 \end{pmatrix}. \tag{C.21}$$

Case B: $\lambda_1^2 = \frac{N_1^2+N_2^2}{2} - l^2$, which has a degeneracy $2l$, with the states given by

$$\begin{pmatrix} -i\sqrt{\frac{l-m}{l+m+1}} Y_{lm} \\ Y_{l,m+1} \end{pmatrix} \text{ for } m = -l, \ldots, l-1. \tag{C.22}$$

One can check that the total number of states in cases A and B is equal to $2N_1N_2$ as expected.

Next we solve for the eigenvectors of the operator $\{\Delta_1, \cdot\}$ by making linear combinations of the those of $\{\Delta_1, \{\Delta_1, \cdot\}\}$ found above. To do this we have to make extensive use of the formula (B.4). We find

- Take the extremal value $l = \frac{N_1+N_2}{2} - 1$ in the case A above, all the $2l + 2 = N_1 + N_2$ eigenvectors of $\{\Delta_1, \{\Delta_1, \cdot\}\}$ worked out above are automatically eigenvectors of $\{\Delta_1, \cdot\}$, with eigenvalue being $\lambda_1 = \frac{N_2-N_1}{2}$.

- Take the extremal value $l = \frac{|N_1-N_2|}{2}$ in case B, all the $2l = |N_1 - N_2|$ eigenvectors of $\{\Delta_1, \{\Delta_1, \cdot\}\}$ worked out above are automatically eigenvectors of $\{\Delta_1, \cdot\}$, with eigenvalue being $\lambda_1 = \left(\frac{N_1+N_2}{2}\right)\left(\frac{N_1-N_2}{|N_1-N_2|}\right)$.

- For generic values of l, one has to choose a state from case A and a state from case B and linearly combine them. The result is: $l = \frac{|N_1-N_2|}{2}, \ldots, \frac{N_1+N_2}{2} - 2$, for any given l, the eigenvalues of $\{\Delta_1, \cdot\}$ are $\lambda_1 = \pm\sqrt{\frac{N_1^2+N_2^2}{2} - (l+1)^2}$ with a degeneracy $2l + 2$. We omit the expressions of the eigenvectors here since those are long and won't be needed anyway.

As one can check, the total number of the above eigenvectors for $\{\Delta_1, \cdot\}$ is

$$N_1 + N_2 + |N_1 - N_2| + \sum_{l=\frac{|N_1-N_2|}{2}}^{\frac{N_1+N_2}{2}-2} 2(2l+2) = 2N_1N_2, \tag{C.23}$$

as expected. This completes our solving the eigenproblem (C.14).

The eigen-equation (C.15) which we write as $\{\Delta_2, \theta_2\} = \lambda_2\theta_2$ with $\Delta_2 \equiv (\tau_2 J^1 + \tau_1 J^2 + \tau_3 J^3)$ is now easy to solve. One can readily check that if $\theta_1 = \binom{\beta}{\eta}$ satisfies $\{\Delta_1, \theta_1\} = \lambda_1\theta_1$, then $\theta_2 = \binom{-\eta}{\beta}$ satisfies $\{\Delta_2, \theta_2\} = \lambda_2\theta_2$ with $\lambda_2 = \lambda_1$, i.e. Δ_2 has the same eigenvalues and degeneracies as Δ_1 does.

Let us summarize the off-diagonal fermionic fluctuation χ's mass spectrum. The mass spectrum of χ_+ (which has a total number of $8N_1N_2$ real d.o.f.'s) is

$$m_{\chi_+} = \sqrt{\frac{v}{\mu} + \frac{x^a x^a}{\mu^2} + \left(\frac{\lambda}{3} - \frac{1}{4}\right)^2}, \tag{C.24}$$

with

$$\lambda = \frac{N_2 - N_1}{2}, \qquad \text{degeneracy: } 4(N_1 + N_2),$$

$$\lambda = \frac{N_1 + N_2}{2}\left(\frac{N_1 - N_2}{|N_1 - N_2|}\right), \qquad \text{degeneracy: } 4|N_1 - N_2|,$$

$$l = \frac{|N_1 - N_2|}{2}, \ldots, \frac{N_1 + N_2}{2} - 2,$$

$$\lambda = \pm\sqrt{\frac{N_1^2 + N_2^2}{2} - (l+1)^2}, \qquad \text{degeneracy: } 8l + 8, \text{ for each sign.} \tag{C.25}$$

The mass spectrum of χ_- (which also has a total number of $8N_1N_2$ real d.o.f.'s) is obtained from that of χ_+ by simply changing v to $-v$.

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